Differential Linear Regression Model For Frequency Estimation

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Abstract – *This paper gives an account of how the frequency estimation problem can be reformulated and solved as a linear regression problem. The resulting solution is shown to be prone to noise due to differentiation-based signal noise-amplification inherent in the linear regression model formulation. This drawback is mitigated by using the signal's autocorrelation function rather than the signal itself.*

***Key Words*: Autocorrelation, Linear Regression, Noise Amplification, Frequency Estimation, Autoregression.**

1. INTRODUCTION

The frequency estimation problem is one of the actively researched problems in various fields which make use of signal and data processing. The techniques used for frequency estimation can largely be classified as time/spatial domain-based methods [1], [2], [3], [4], [5], [6] and frequency domain-based methods [3], [4], [7]. In many time-domain formulations, the problem is posed as a nonlinear problem [8], [9]. Frequency estimation can be considered as a special case of a more general problem of parameter and state estimation often encountered in control systems design and signal processing. This problem is relevant in many fields of Science including Physics [10], [11], Engineering [11], [12], Finance and Economics [13], and many others.

In this paper, we present one way to model the frequency estimation problem as a linear regression problem with a closed-form solution. The approach is based on shifting the problem from how the signal output is related to time or frequency to how the signal output is related to itself either sequentially (i.e. autoregressively) or differentially (i.e. satisfying a particular time-invariant differential equation). In both cases, there are time-invariant coefficients that can be estimated in a linear regression problem formulation.

The rest of this paper is organized as follows. Section 2 presents a general damped second-order oscillatory system model from which a free/unforced oscillation response model is adopted as a baseline model for frequency estimation. Section 3 gives a detailed account of formulating the frequency estimation problem as a linear regression problem. It is shown here that either the raw data or the autocorrelation function of the data can be used for frequency estimation, with the drawbacks of each option mentioned. Section 4 presents simulation results for the models in Section 3 and gives a brief discussion on these results. Section 5 concludes this work with a summary of major findings and some remarks.

2. A DAMPED OSCILLATOR MODEL

2.1 General Second-order System

A general causal second-order system can be with input $u(t)$ and output $y(t)$ can be modeled as shown below [12],

$$y'' + \frac{\omega_n}{Q} y' + \omega_n^2 y = au'' + bu' + c\omega_n^2 u \qquad (1)$$

with ω_n is the natural frequency, Q is the quality factor (Q -factor), a measure of how damped the system is or the selectivity in the frequency of the system. The low-frequency gain c controls the low pass filtering behaviour independent of resonance while a and b are the gains associated with high pass filtering and bandpass filtering respectively. The resulting transfer function takes the following form [12],

$$G(s) = \frac{Y(s)}{U(s)} = \frac{as^2 + bs + c\omega_n^2}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2} \qquad (2)$$

with s as the Laplace parameter in the frequency domain. The transfer function model is very useful for frequency domain analysis and design such as pole placement is done in control theory and filter design.

2.2 Free/Unforced Damped Response

An unforced/undriven harmonic oscillation can be modeled by the homogeneous part of equation (1) as shown below,

$$y'' + \frac{\omega_n}{Q} y' + \omega_n^2 y = 0 \qquad (3)$$

with its solution taking the following form,

$$y(t) = Ae^{-\xi\omega_n t} cos\left(\omega_n t\sqrt{1-\xi^2} + \theta\right) + b \qquad (4)$$

where A is the amplitude, b is the signal offset, θ is the phase offset and the parameter $\xi = (2Q)^{-1}$ is the damping ratio.

3. REGRESSION-BASED FREQUENCY ESTIMATION

3.1 Signal Data-based Linear Regression Model

Given a data $y_i(t_i)$ of size N (i.e. $i = 1, 2, 3, \dots, N$) from an oscillatory process, we can formulate the problem of frequency estimation as a linear regression problem by adopting equation (3) as part of the regression cost function shown below,

$$J(\alpha, \beta) = \sum_{i=1}^{N} (y_i'' + \alpha y_i' + \beta y_i)^2 \qquad (5)$$

with $\alpha = \omega_n Q^{-1}$ and $\beta = \omega_n^2$. The resulting optimal solution to this regression problem is given by,

$$\alpha = \frac{\sum_{i=1}^{N} y_i y_i' \sum_{i=1}^{N} y_i y_i'' - \sum_{i=1}^{N} y_i^2 \sum_{i=1}^{N} y_i' y_i''}{\sum_{i=1}^{N} y_i^2 \sum_{i=1}^{N} {y_i'}^2 - \left(\sum_{i=1}^{N} y_i y_i'\right)^2} \qquad (6)$$

$$\beta = \frac{\sum_{i=1}^{N} y_i y_i' \sum_{i=1}^{N} y_i' y_i'' - \sum_{i=1}^{N} {y_i'}^2 \sum_{i=1}^{N} y_i y_i''}{\sum_{i=1}^{N} y_i^2 \sum_{i=1}^{N} {y_i'}^2 - \left(\sum_{i=1}^{N} y_i y_i'\right)^2} \qquad (7)$$

From which the frequency and quality factor are estimated as,

$$\omega_n = \sqrt{\beta} \qquad (8)$$

$$Q = \omega_n \alpha^{-1} \qquad (9)$$

In this data-based formulation, this model does not perform well in noisy data as explained in the next section.

3.2 Autocorrelation-based Linear Regression Model

Looking at the cost function in equation (5), one sees that it involves derivatives of the data and this differentiation can amplify the high-frequency noise content in the data and lead to inaccurate frequency estimation. One way to counter this drawback is by replacing the data $y_i(t_i)$ with its autocorrelation function $z_i(t_i)$ in the cost function above. The resulting optimal solution takes the same form as the one derived above except for y_i being replaced by z_i everywhere as shown below,

$$\alpha = \frac{\sum_{i=1}^{N} z_i z_i' \sum_{i=1}^{N} z_i z_i'' - \sum_{i=1}^{N} z_i^2 \sum_{i=1}^{N} z_i' z_i''}{\sum_{i=1}^{N} z_i^2 \sum_{i=1}^{N} {z_i'}^2 - \left(\sum_{i=1}^{N} z_i z_i'\right)^2} \qquad (10)$$

$$\beta = \frac{\sum_{i=1}^{N} z_i z_i' \sum_{i=1}^{N} z_i' z_i'' - \sum_{i=1}^{N} {z_i'}^2 \sum_{i=1}^{N} z_i z_i''}{\sum_{i=1}^{N} z_i^2 \sum_{i=1}^{N} {z_i'}^2 - \left(\sum_{i=1}^{N} z_i z_i'\right)^2} \qquad (11)$$

Introducing the autocorrelation function has a summing and an averaging effect which can be inferred as integration and filtering. This integrating effect counters and compensates for the derivative operations in the cost function thus improving the noise-rejection feature of the linear regression model. The autocorrelation operation can be performed on the autocorrelation function itself to further improve the noise-rejection feature. Since the frequency of the data is invariant to autocorrelation operation, the frequency estimation will not be affected by using the autocorrelation function over the data itself. However, the quality factor is not invariant under autocorrelation operation, hence it is subject to change thus not necessarily giving a good estimate of the quality factor inherent in the data itself.

4. SIMULATION RESULTS & DISCUSSION

This section presents the simulation of the proposed linear regression model for a case of noiseless data and a case of noisy data. Below we begin with a case of noiseless data.

4.1 Frequency Estimation With Noiseless Data

4.1.1 Signal Data-based Frequency Estimation

Fig. 1 below shows the plot of data from which the frequency should be estimated with the regression model described in the previous section. The data was generated with the following parameters corresponding to equation(4); $A = 5\,V$, $b = 7\,V$, $\theta = 0.25\pi\,rads$, $\omega_n = 3\pi\,rad/s = 9.4248\,rad/s$, $\xi = (15\pi)^{-1} = 0.02122$.

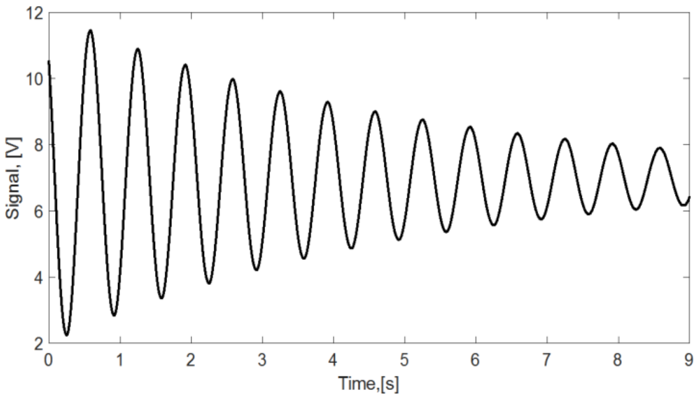

Fig. 1 Noiseless data.

Fig. 2 below shows the same data shown in Fig. 1 but with a constant offset estimate $b = N^{-1} \sum_{i=1}^{N} y_i \approx 6.9527\,V$ removed, resulting in zero-mean data.

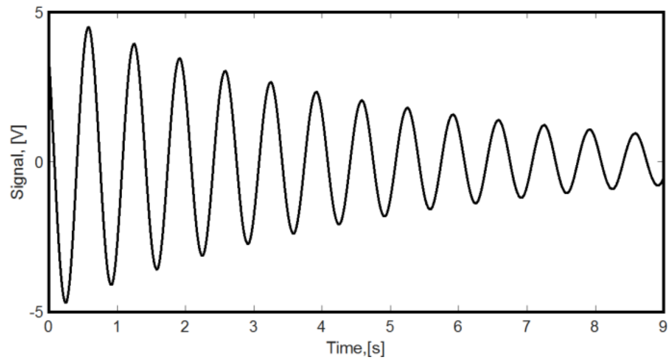

Fig. 2 Zero-mean noiseless data.

The estimated offset $b \approx 6.9527$ is 0.7% lower than the exact value of $b = 7.00$. Fig. 3 below shows the first derivatives of the zero-mean data. The effect of noise-amplification is not evident since the underlying is noise-free.

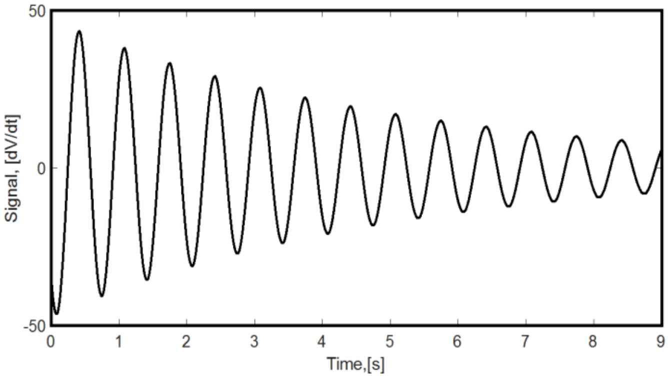

Fig. 3 The first derivatives of zero-mean data.

Fig. 4 below shows the second derivatives of the zero-mean data which also looks clean due to the underlying data being noise-free.

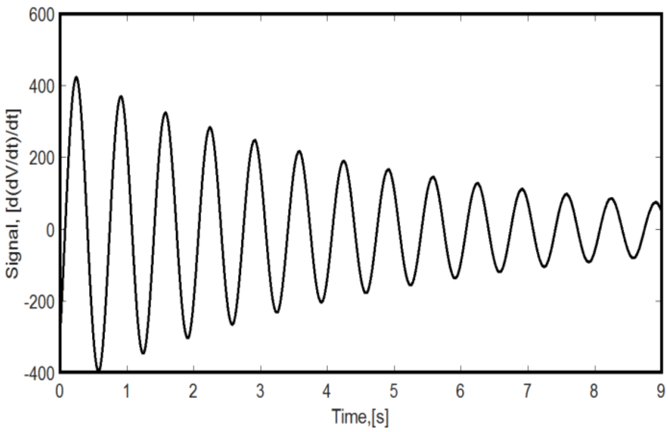

Fig. 4 The second derivatives of zero-mean data.

Based on the zero-mean data and its first two orders of differentiation, the linear regression model explained in the previous section was used to estimate the frequency and quality factor.

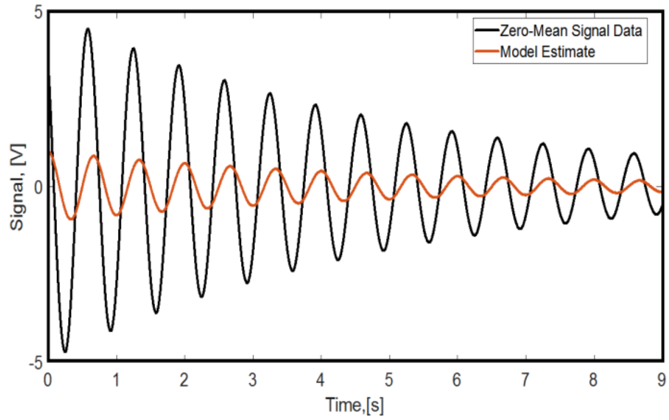

Fig. 5 Frequency and Q-factor estimates simulated.

Fig. 5 above shows the plot of linear regression model-based estimate superimposed on the zero-mean data. The recovered frequency and damping ratio estimates are,

$$\omega_n \approx \sqrt{\beta} = 9.4260\, rad/s \qquad (12)$$

$$\xi \approx \frac{\alpha}{2\omega_n} = \frac{0.2031}{9.4260} = 0.02155 \qquad (13)$$

with the estimated frequency being off by 0.013% from the exact value and the estimated damping ratio being off by 1.6% from the exact value. The amplitude and phase offset are not estimated in this work since the main focus is on estimating the frequency.

4.1.2 Autocorrelation-based Frequency Estimation

Fig. 6 below shows the plot of the autocorrelation function of the zero-mean data superimposed on the zero-mean data plot. The same frequency estimation will be carried out using the autocorrelation function and its derivatives here. The idea is to compare these two ways of estimating frequency under both noise-free and noise-infested data cases.

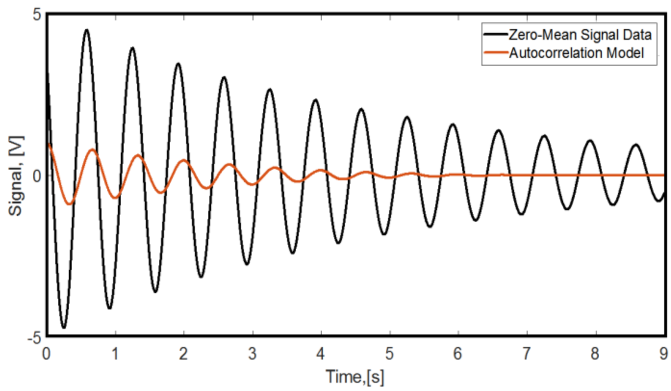

Fig. 6 Zero-mean data and its autocorrelation function.

Fig. 7 below shows the first-order derivatives of the autocorrelation function shown in Fig. 6 above.

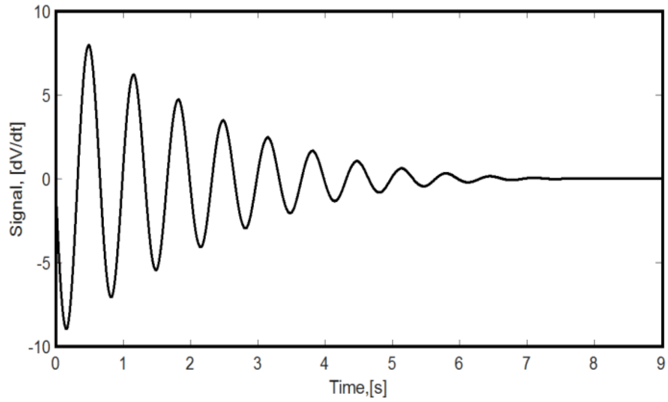

Fig. 7 Autocorrelation function's first derivatives.

Fig. 8 below shows the second derivatives of the autocorrelation function shown in Fig. 6 above.

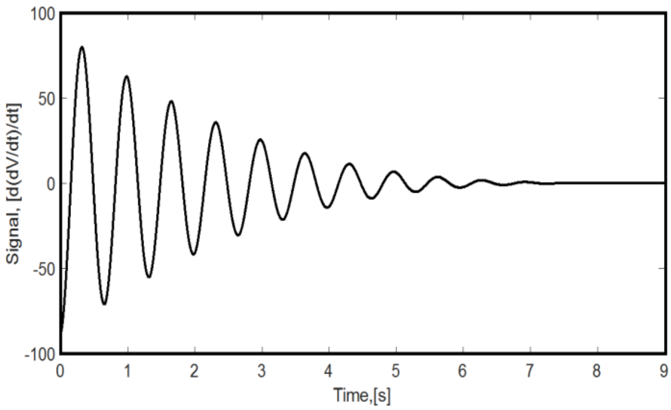

Fig. 8 Autocorrelation function's second derivatives.

Based on the autocorrelation function and its first two orders of differentiation, the linear regression model explained in the previous section was used to estimate the frequency of the zero-mean data and the quality factor of the autocorrelation function itself.

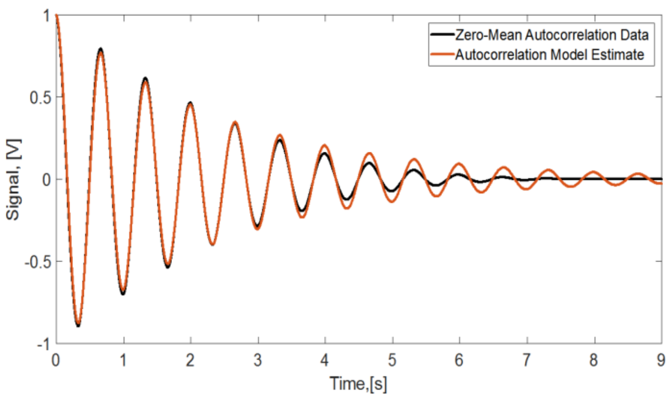

Fig. 9 Frequency and Q-factor estimates simulated.

Fig. 9 above shows the plot of the resulting linear regression model-based estimate superimposed on the autocorrelation function. The recovered frequency and damping ratio estimates are,

$$\omega_n \approx \sqrt{\beta} = 9.4465\, rad/s \tag{14}$$

$$\xi \approx \frac{\alpha}{2\omega_n} = \frac{0.3976}{9.4465} = 0.04209 \tag{15}$$

with the estimated frequency being off by 0.23% from the exact value. The estimation error here is about 18 times higher than in the previous case of the signal data-based estimation approach. The damping factor here is not related to our data of interest but rather related to its autocorrelation function. It can be observed qualitatively from the plot in Fig. 9 that the estimated damping ratio is smaller than that of the autocorrelation function based on the slower damping effect on the estimate compared to that seen on the autocorrelation function. The amplitude and phase offset are not estimated in this work since the main focus is on estimating the frequency.

4.2 Frequency Estimation With Noisy Data

4.2.1 Signal Data-based Frequency Estimation

In this second case or experiment, we look at the two frequency estimation approaches under the noise-infested signal data. The signal data has the same parameter setup as in the previous case except that here the is also an additive zero-mean white noise with an amplitude of 10 V.

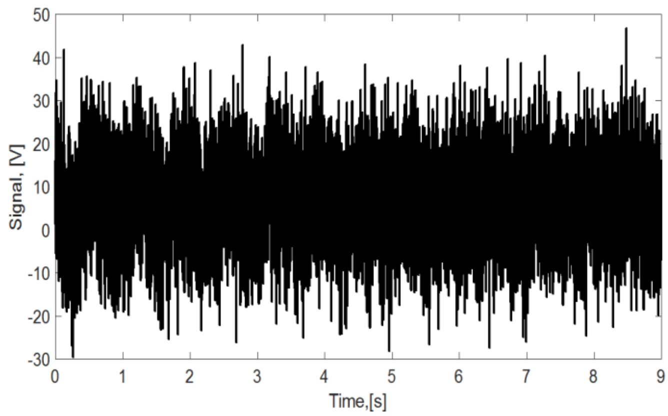

Fig. 10 The plot of noisy data.

Fig. 10 above shows the plot of this noise signal data based on the model in equation (4). As before Fig. 11 below shows the zero-mean version of the noise data.

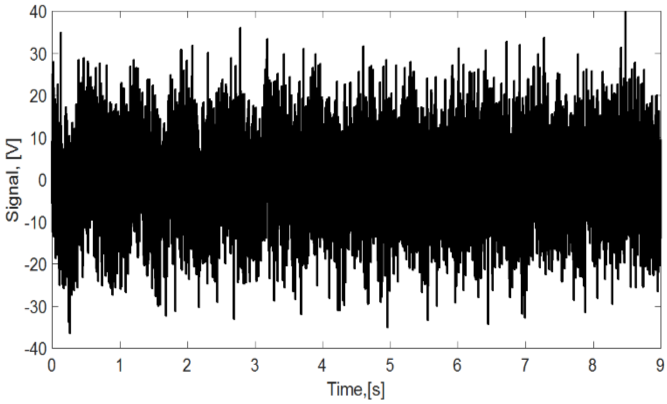

Fig. 11 Zero-mean noisy data.

The offset b was estimated as before and found to be $6.9306\,V$ which is 1% off from the exact value. Fig. 12 below shows the first-order derivatives of the noisy data. It becomes clear here that the noise inherent in the data gets amplified by taking the derivatives as one looks at the magnitude.

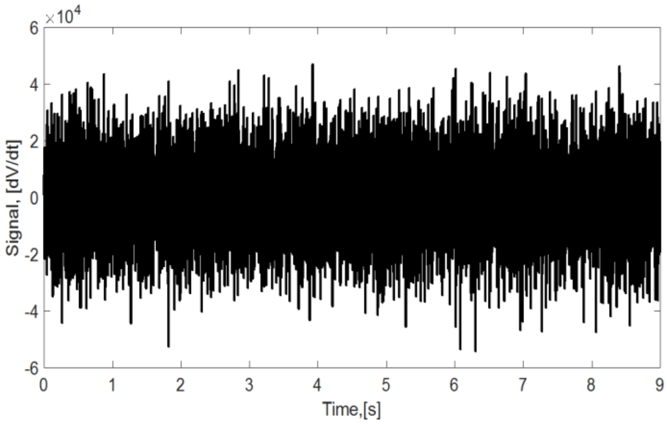

Fig. 12 The first derivatives of zero-mean data.

Fig. 13 below shows the second-order derivatives of the signal data. This shows even further noise amplification.

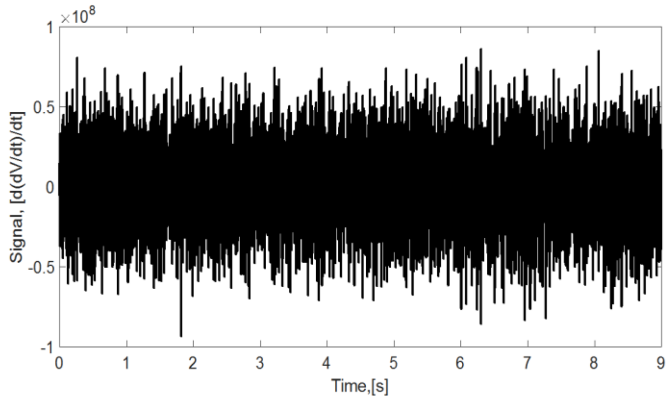

Fig. 13 The second derivatives of zero-mean data.

Fig. 14 below shows the data-based frequency estimation model plot superimposed on the zero-mean signal data plot.

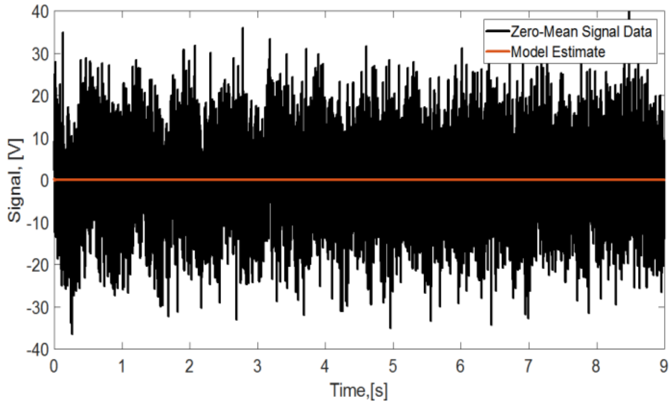

Fig. 14 Frequency and Q-factor estimate simulated.

It is unclear what to make of this estimate plot other than saying it has failed to reasonably estimate both the frequency and damping ratio. This is because the resulting frequency and damping ratio estimates are given as,

$$\omega_n \approx \sqrt{\beta} = 947.4746\, rad/s \qquad (16)$$

$$\xi \approx \frac{\alpha}{2\omega_n} = \frac{521.4495}{947.4746} = 0.55 \qquad (17)$$

which are both out by more than 100%. It was expected that the data-based estimation model will fail in the presence of noise due to the inherent noise-amplification in the model formulation. Next, we look at how the autocorrelation based estimation model handle the same problem under the same noise conditions,

4.2.2 Autocorrelation-based Frequency Estimation

Fig. 15 below shows the plot of the autocorrelation of the noisy data superimposed o the plot of the noisy data itself.

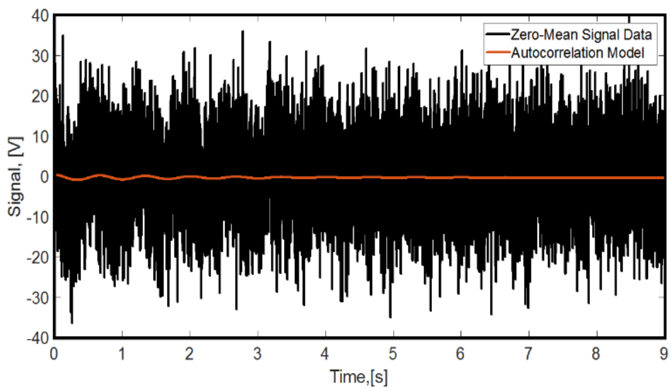

Fig. 15 Data and its autocorrelation function.

Fig. 16 shows the first derivatives of the autocorrelation function of the noisy data. It can be seen that both Fig 15 and Fig 16 plots show a significant noise-rejection or noise-attenuation due to the integrating and/or averaging effect inherent in the autocorrelation function operation.

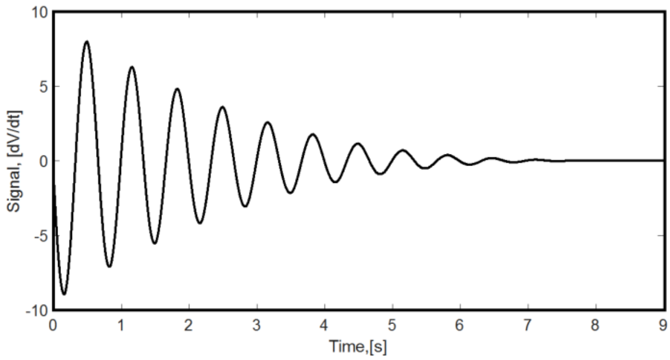

Fig. 16 Autocorrelation function's first derivatives.

Fig. 17 shows the second-order derivatives of the autocorrelation function of the noisy data, which also look fairly clean.

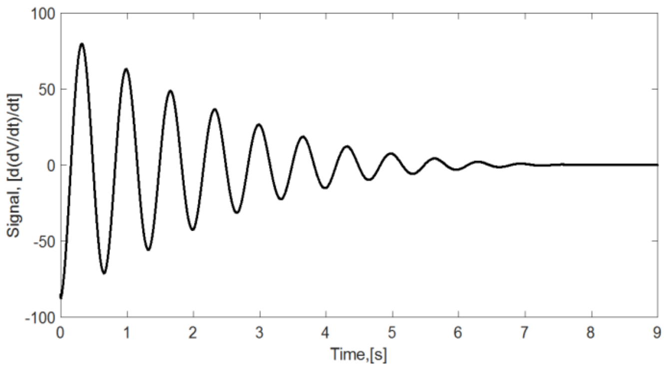

Fig. 17 Autocorrelation function's second derivatives.

Based on the autocorrelation function and its first two orders of differentiation, the linear regression model explained in the previous section was used to estimate the frequency of the zero-mean data and the quality factor of the autocorrelation function itself.

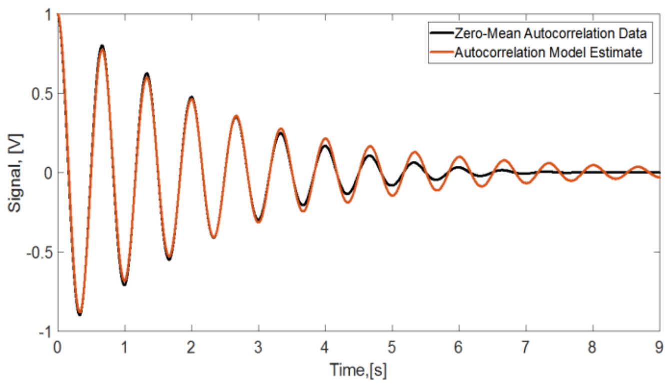

Fig. 18 Frequency and Q-factor estimate simulated.

Fig. 18 above shows the plot of the resulting linear regression model-based estimate superimposed on the autocorrelation function. The recovered frequency and damping ratio estimates are,

$$\omega_n \approx \sqrt{\beta} = 9.4426\, rad/s \tag{18}$$

$$\xi \approx \frac{\alpha}{2\omega_n} = \frac{0.3890}{9.4426} = 0.04120 \tag{19}$$

with the estimated frequency being off by 0.19% from the exact value. The frequency estimation error here is still very good even under the condition of noise-infested signal data. This is indicative of the high robustness of the autocorrelation-based linear regression model for estimating frequency. The estimated damping factor can still be observed qualitatively from the plot in Fig. 9 to be smaller than that of the autocorrelation function based on the slower damping effect on the estimate compared to that seen on the autocorrelation function. The amplitude and phase offset are not estimated in this work since the main focus is on estimating the frequency.

5. CONCLUSIONS

In this work, we have successfully shown how the frequency estimation problem can be posed as a linear regression problem based on the differential relationship of the oscillatory data. It was shown that this model suffers from noise amplification due to the differential operations inherent in the linear regression formulation. This drawback was successfully mitigated by using the autocorrelation function of the data instead of the data itself. This noise mitigation also brought a problem of not being able to obtain a good estimate of the quality factor in case one wishes to estimate both the frequency and quality factor from the data. It would be interesting to investigate the autoregressive relationship of the oscillatory data as the basis for the linear regression model since it is likely to not suffer from noise amplification brought about by the differential operations.

REFERENCES

[1] Ignjatovic A. Frequency estimation using time-domain methods based on robust differential operators. 10th IEEE International Conference on Signal Processing (ICSP), 1-6, 2010. DOI:10.1109/ICOSP.2010.5657057

[2] Seyedi H., Sanaye-Pasand M. A new time-domain based power system frequency estimation algorithm. European Transactions on Electrical Power. 1-16, 2011, DOI:10.1002/etep.572

[3] Kootsookos P. A Review of the Frequency Estimation and Tracking Problems. Technical Report, CRC for Robust and Adaptive Systems, 1999.

[4] Liao Y. Phase and Frequency Estimation: High-Accuracy and Low-Complexity Techniques. Masters thesis, Worcester Polytechnic Institute, 2011.

[5] Fowler M. L. Phase-based Frequency Estimation: A Review. Digital Signal Processing. 12(4): 590-615, 2002.

[6] Priestly M. B. Estimation of Frequencies. In: Robinson P.M., Rosenblatt M. (eds) Athens Conference on Applied Probability and Time Series Analysis. Lecture Notes in Statistics, vol 115. Springer, New York, NY. https://doi.org/10.1007/978-1-4612-2412-9_23, 1996.

[7] Jensen R. J., Neilsen J. K., Christensen M. G., Jensen S. H. On Frequency Domain Models for TDOA Estimation. DOI: 10.1109/ICASSP.2015.7177922, 2015.

[8] Besson O., Stoica P. Nonlinear Least-Squares Approach to Frequency Estimation and Detection for Sinusoidal Signals with Arbitrary Envelope. Digital Signal Processing, 9(1): 45-56, 1999.

[9] Sen Z., Jiulong W., Dengyun Y., Frequency Estimation using Nonlinear Least Squares and S-Transform. 3rd International Congress on Image and Signal Processing, pp. 138-142, DOI:10.1109/CISP.2010.5646366, 2010.

[10] Matjelo N. J., Khanyile N., Uys H. Demonstration Of Rabi-Flops With Ytterbium 171 Trapped-Ion Qubits. International Journal of Scientific and Research Publications (IJSRP), 11(7), June 2021.

[11] Matjelo N. J., Khanyile N., Uys H. Design, Implementation, and Characterization of Helical Resonator for Ion Trapping Application. International Research Journal of Modernization in Engineering Technology and Science (IRJMETS), 3(5):1285-1295, 2021.

[12] Matjelo N. J., Khanyile N., Uys H. Design, Implementation, and Characterization of RLC Resonator for Ion Trapping Application. International Research Journal of Modernization in Engineering Technology and Science (IRJMETS), 3(5):1879-1888, 2021.

[13] Sandoval L. Jr., Franca I. P. Shocks in Financial Markets, Price expectation, and Damped Harmonic Oscillators. arXiv:1103.1992v2 [q-fin.GN], 2011.