Applications of Differential Transform Method

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Abstract: In this paper, the initial value problem is solved by differential transform method. The proposed method is promising to a broad class of linear and non-linear problems. The result of differential transform method is in good agreement with those obtained by using already existing ones. The differential transform method appeared to be effective, reliable, easy and flexible for finding the solutions for such type of initial value problems. DTM is an analytical & numerical method for solving a wide variety of numerical differential equations and usually gets the solution in series form.

Keywords: Initial value problems, Linear, Non-linear, Differential transform, Numerical method.

1. Introduction

Many problems of different fields like engineering, physics and geology are described by ordinary or partial differential equations with appropriate value problems. The differential transform method is very effective and powerful tool for solving various kinds of differential equations. Zhou introduced the basic idea of DTM in 1980, to solve linear and non-linear initial value problem that applies in electrical circuits. DTM is an iterative procedure for obtaining analytical Taylor Series solution of differential equations. The Taylor series method computationally takes long time for large orders. The main advantage of this method is that this can be applied directly to differential equations without recurring linearization, discretization.

2. Differential transform method (DTM):

Differential transform of the function y(x), is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \qquad \text{..........(1)}$$

and inverse differential transform of Y(k) is defined by:

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \qquad \text{..........(2)}$$

Based on the above definitions, the fundamental operations of differential transform method as shown below:

3. One dimensional differential transformation:

3.1. If y(x) = w(x) ± z(x), then, Y (k) = W (k) ± Z (k)

3.2. If y(x) = c w(x), then, Y (k) = c W (k), where "c" is constant.

3.3. If y(x) = (dw(x))/dx, then, Y (k) = (k+1) W (k).

3.4. If y(x) = (d^m w(x))/ (dx^m), then, Y (k) = (k+1) (k+2) -------- (k+m) W (k+m).

3.5. If y(x) = 1, then Y (k) =δ (k).

3.6. If y(x) = x, then Y (k) = δ (k-1).

3.7. If y(x) = w(x) z(x), then, Y (k) =$\sum_{m=0}^{k} W(k)\, z(k-m)$.

3.8. If y(x) =x^m, then Y (k) = δ(k-m) = 1, if k=m

= 0, if k$\neq m$

Here m is non-negative integer

3.9. If y(x) =e^{ax}, then Y (k) = $\frac{e^{ak}}{k!}$, Where a is constant.

3.10. If y(x) =$(1+x)^m$, then Y (k) = $\frac{\mathrm{m\,(m-1)\,(m-2)\ldots\ldots\ldots(m-k+1)}}{k!}$

3.11. If y(x) = sin (wx + α), then Y (k) = $\frac{w^k}{k!}$ sin ($\frac{k\pi}{2}$ + α), Here α, w are constant.

3.12. If y(x) = cos (wx + α), then Y (k) = $\frac{w^k}{k!}$ cos ($\frac{k\pi}{2}$ + α), Here α, w are constant.

4. Numerical Applications:

In this section, we will apply Differential transform method to some initial value problem:

4.1. Problem 1:

Consider the equation with initial value

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^4 \;,\quad y(1) = \frac{2}{3},\; y'(1) = \frac{7}{3} \quad \ldots\ldots\ldots\ldots(3)$$

Here we are taking transformation x =e^t, $\log x$ = t
Therefor equation (3) becomes with initial value

$$\frac{d^2y}{dx^2} - y = e^{4t} \;,\quad y(0) = \frac{2}{3},\, y'(0) = \frac{7}{3}\ldots\ldots\ldots\ldots(4)$$

Now, we apply differential transform method, on equation (4), gives:

$$(k+1)(k+2)Y(k+2) - Y(k) = \frac{4^k}{k!}\ldots\ldots\ldots\ldots(5)$$

$$\left.\begin{array}{l} \text{Thus, } Y(k+2) = \frac{1}{(k+1)(k+2)}\left[\frac{4^k}{k!} + Y(k)\right] \\ \text{With the conditions } y(0)=\frac{2}{3},\; y(1) = \frac{7}{3} \end{array}\right\} \ldots\ldots\ldots\ldots(6)$$

When k=0, in equation (6), then

$$Y(2) = \frac{1}{(0+1)(0+2)}\left[\frac{4^0}{0!} + Y(0)\right] = \frac{1}{2}\left[1 + \frac{2}{3}\right] = \frac{5}{6}$$

When k = 1, in equation (6), then

$$Y(3) = \frac{1}{(1+1)(1+2)}\left[\frac{4^1}{1!} + Y(1)\right] = \frac{1}{(2)(3)}\left[4 + \frac{7}{3}\right] = \frac{19}{18} \quad \ldots\ldots (8)$$

When k = 2, in equation (6), then

$$Y(4) = \frac{1}{(2+1)(2+2)}\left[\frac{4^2}{2!} + Y(2)\right] = \frac{1}{(3)(4)}\left[\frac{16}{2} + \frac{5}{6}\right] = \frac{53}{72} \ldots\ldots (9)$$

We have $Y(2) = \frac{5}{6}$, $Y(3) = \frac{19}{18}$, $Y(4) = \frac{53}{72}$, ………………

Then, we have following solution to initial value problem:

$$y(t) = \sum_{k=0}^{\infty} t^k\, Y(k) \quad \ldots\ldots (10)$$

$$= t^0 Y(0) + t^1 Y(1) + t^2 Y(2) + t^3 Y(3) + t^4 Y(4) + o\,(t^5)$$

i.e. $y(t) = \frac{2}{3} + t\left(\frac{7}{3}\right) + t^2\left(\frac{5}{6}\right) + t^3\left(\frac{19}{18}\right) + t^4\left(\frac{53}{72}\right) + \ldots\ldots$

$$y(t) = \frac{2}{3} + \frac{7}{3}t + \frac{5}{6}t^2 + \frac{19}{18}t^3 + \frac{53}{72}t^4 + O(t^5) + \ldots\ldots (11)$$

4.2. Problem 2:

Now, consider the following with initial value problem:

$$\left.\begin{aligned} & x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + y^2 = (\log x)^2 \\ & \text{With } y(1) = 0,\ y'(1) = 1 \end{aligned}\right\} \quad \ldots\ldots (12)$$

Taking transformation $x = e^t$, $\log x = t$, the equation (12) becomes,

$$\left.\begin{aligned} & \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y^2 = (t)^2 \\ & \text{With } y(0) = 0,\ y'(0) = 1 \end{aligned}\right\} \quad \ldots\ldots (13)$$

Now, we apply DTM method on equation (13)

$$(k+1)(k+2)Y(k+2) + 4(k+1)Y(k+1) + \sum_{m=0}^{k} Y(k)\,Y(k-m) = \delta(k-2) \quad \ldots\ldots (14)$$

Thus,

$$\left.\begin{aligned} & Y(k+2) = \frac{1}{(k+1)(k+2)}\left[-4(k+1)Y(k+1) - \sum_{m=0}^{k} y(k)\,y(k-m) + \delta(k-2)\right] \quad \ldots\ldots (15) \\ & \text{With following conditions: } y(0) = 0,\ y(1) = 1 \end{aligned}\right\}$$

Now, we find Y (2), Y (3), Y (4) …………………… If we put k=0 in the equation (15), then

$$Y(2) = \frac{1}{(0+1)(0+2)}\left[-4(0+1)Y(0+1) - \sum_{m=0}^{0} y(0)\,y(0-m) + 0\right]$$

$= \frac{1}{2}[\,(-4)\,(1) - 0 + 0\,] \;=\; -2$ ……………………………………………………………(16)

If we put k=1 in equation (15), then

$Y(3)=\frac{1}{(1+1)(1+2)}[\,-4\,(2)Y(2) - Y(1)Y(1) - Y(1)Y(0) + 0]=\frac{1}{6}[\,16 - 1 - 0 + 0\,]=\frac{15}{6}$ ………(17)

If we put k=2 in equation (15), then

$Y\,(4) = \frac{1}{(2+1)(2+2)}[\,-4\,(3)Y(3) - Y(2)Y(2) - Y(2)Y(1) - Y(2)Y(0) + 1]=$

$\frac{1}{12}\left[-12\,\left(\frac{15}{6}\right)(-2)(-2) - (-2)(1) - (-2)\,(0) + 1\,\right]=\frac{1}{12}[\,-30 - 4 + 2 + 0 + 1\,]=\frac{-31}{12}$ ……(18)

Therefore, we have:

$Y(2) = -2, Y(3) = \frac{15}{6}, \quad Y(4) = \frac{-31}{12}$…………

Thus, its solution is:

$y(t) = \sum_{k=0}^{\infty} t^k\, Y(k)$……………………………………………………………………………(19)

$y(t) = t^0\,Y\,(0) + t^1 Y\,(1) + t^2\,Y(2) + t^3\,Y(3) + t^4\,Y\,(4) + O(t^5) +$ ……….

$y(t) = t - 2t^2 + \frac{15}{6}t^3 - \frac{13}{12}t^4 + O(t^5) +$ …………………… ……………………………………(20)

4.3. Problem 3:

Consider the equation with initial value problem:

$$\left.\begin{aligned} &x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 2y = x \\ &\text{With initial conditions: } y(1) = 1,\; y'(1) = -2 \end{aligned}\right\} \qquad \text{……………………………………(21)}$$

Taking transformation $x = e^t$, $\log x = t$, the equation (21) becomes,

$$\left.\begin{aligned} &\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2y = e^t \\ &\text{With } y\,(0) = 1,\; y'(0) = -2 \end{aligned}\right\} \qquad \text{……………………………………………………………(22)}$$

Now, we apply DTM method on equation (22)

$(k+1)\,(k+2)\,Y(k+2) \;+2(k+1)\,Y(k+1) \;-2\,Y(k)= \frac{1^k}{k!}$

Thus,

$$\left.\begin{aligned} &Y(k+2) = \frac{1}{(k+1)(k+2)}\left[-2(k+2)Y(k+1) + 2Y(k) + \frac{1}{k!}\right] \\ &\text{And the initial conditions are: } y(0) = 0,\ y(1) = -2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(23)$$

Substituting these values at k =0, 1, 2, 3………….. in above equation

$$Y(2) = \frac{1}{(0+1)(0+2)}\left[-2(1)Y(1) + 2Y(0) + \frac{1}{0!}\right] = \frac{1}{2}\left[-2(-2) + 2(1) + 1\right] = \frac{7}{2} \quad \ldots\ldots\ldots(24)$$

$$Y(3) = \frac{1}{(1+1)(1+2)}\left[(-2)(2)\left(\frac{7}{2}\right) + (2)(-2) + 1\right] = \frac{-17}{6} \ldots\ldots\ldots(25)$$

$$Y(4) = \frac{1}{(2+1)(2+2)}\left[-2(2+1)Y(3) + 2Y(2) + \frac{1}{2!}\right] = \frac{1}{12}\left[-6\left(\frac{-17}{6}\right) + 2\left(\frac{7}{2}\right) + \frac{1}{2}\right] = \frac{49}{24} \ldots\ldots\ldots(26)$$

$$Y(5) = \frac{1}{(3+1)(3+2)}\left[-2(3+1)Y(4) + 2Y(3) + \frac{1}{3!}\right] = \frac{1}{20}\left[-8\left(\frac{49}{24}\right) + 2\left(\frac{-17}{6}\right) + \frac{1}{6}\right] = \frac{-131}{120} \ldots\ldots\ldots(27)$$

We obtained the close form solution up to N=5, so, we have:

$$Y(2) = \frac{7}{2},\quad Y(3) = \frac{-17}{6},\quad Y(4) = \frac{49}{24},\ Y(5) = \frac{-131}{120} \ldots\ldots$$

Its solution is:

$$y(t) = \sum_{k=0}^{\infty} t^k\, Y(k)$$

$$= t^0Y(0) + t^1Y(1) + t^2Y(2) + t^3Y(3) + t^4Y(4) + t^5Y(5) + O(t^6) + \ldots\ldots\ldots(28)$$

$$= 1 + t(-2) + t^2\left(\frac{7}{2}\right) + t^3\left(\frac{-17}{6}\right) + t^4\left(\frac{49}{24}\right) + t^5\left(\frac{-131}{120}\right) + O(t^6) + \ldots$$

$$= 1 - 2t + \frac{7}{2}t^2 - \frac{17}{6}t^3 + \frac{49}{24}t^4 - \frac{131}{120}t^5 + O(t^6) - \ldots\ldots\ldots(29)$$

Conclusion

In this paper we extended the applications of differential transform method which is based on Taylor series expansion, to construct analytical approximate solutions of the initial value problems. DTM is simple , easy to use & produce reliable results.

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