COMPLETE MEET HYPERLATTICES

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Abstract: *Hyperlattices are a suitable generalization of an ordinary lattice. Here, we consider about meet hyperlattice and complete meet hyperlattice. Also, we consider some properties of them.*

Key words:

Algebraic hyper structures; hyperlattice; ideals; filers; prime element; Complete hyperlattice.

1. INTRODUCTION

Lattices, especially Boolean algebras, arise naturally in logic, and thus, some of the elementary theory of lattices had been worked out earlier. Nonetheless, there is the connection between modern algebra and lattice theory. Lattices are partially ordered sets in which least upper bounds and greatest lower bounds of any two elements exist. Dedekind discovered that this property may be axiomatized by identities. In this paper, we introduce the concept of complete meet hyperlattices, weak homomorphisms in meet hyperlattice and we investigate their properties.

2. BASIC CONCEPT

First, we consider some basic definitions and well-known facts about hyperlattices.

Let H be a non-empty set. A hyperoperation on H is a map ∘ from H×H to $\wp^*(H)$, the family of non-empty subsets of H. The couple (H, ∘) called a hyper groupoid. For any two non-empty subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b\,; A \circ x = A \circ \{x\}\; and\; x \circ B = \{x\} \circ B$. A hypergroupoid (H,∘) is called a semi hyper group if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$. Moreover, if for any element $a \in H$ equalities $a \circ H = H \circ A = H$ hold, then the pair (H, ∘) is called a hypergroup. A lattice is a partially ordered set L such that for any two elements x, y of L glb $\{x, y\}$ and lub $\{x, y\}$ exist. If L is a lattice, then we define $x \vee y = glb\{x, y\}$ and $x \wedge y = lub\{x, y\}$. This definition is equivalent to the following definition [3]. Let L be a non-empty set with two binary operations $\vee\; and\; \wedge$.Let for all a$a, b, c \in$

L, the following conditions satisfied:

1) $a \wedge a = a\; and\; a \vee a = a$;

2) $a \wedge b = b \wedge a\; and\; a \vee b = b \vee a$;

3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)\; and\; (a \vee b) \vee c = a \vee (b \vee c)$;

4) $(a \wedge b) \vee a = a\;\; and\; (a \vee b) \wedge a = a$;

Then, $(L,\vee,\wedge)$ is a lattice.

Now, we recall the notations of four types of hyperlattices.

JOIN HYPERLATTICE:

Let L be a nonempty set,$\bigvee: L \times L \longrightarrow \wp^*(L)$ be a hyperoperation, and $\wedge: L \times L \rightarrow L$ be an operation. Then, $(L, \bigvee, \wedge)$ is a join hyperlattice if for all $x, y, z \in L$, the following conditions hold:

1)$x \in x \bigvee x$ and $x = x \wedge x$;

2)$x \bigvee (y \vee z) = (x \bigvee y) \bigvee z and\; x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

3)$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;

4)$x \in x \wedge (x \vee y) \cap x \vee (x \wedge y)$.

MEET HYPERLATTICE:

Let L be a nonempty set, $\wedge: L \times L \to \wp^*(L)$ be an operation, and $\vee: L \times L \to L$ be an operation. Then, $(L,\vee,\wedge)$ is a meet hyperlattice if for all $x, y, z \in L$. The following conditions are hold:

1)$x \in x \wedge x$ and $x = x \vee x$;

2)$x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

3)$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;

4)$x \in x \wedge (x \vee y) \cap x \vee (x \wedge y)$.

Total hyperlattice or superlattice.

Let L be both join and meet hyperlattice (which means that $\vee$ and $\wedge$ are both hyper operations). Then, we call L is total hyperlattice.

Example 1: Let $(L,\vee,\wedge)$ be a lattice.

We define $\wedge$ on l as follows

$x \wedge y = \{Z | Z \leq x \wedge y\}$.

Then, $(L,\vee,\wedge)$ is a meet hyperlattice.

Example 2: Let $(L, \leq)$ be a partial order set. We define hyperoperations as follows:

$a \wedge b = \{x \in L: x \geq a, x \geq b\}$ and $a \vee b = \{x \in L: x \leq a, x \leq b\}$. Then, $(L,\vee,\wedge)$ is a total hyperlattice.

3. COMPLETE MEET HYPERLATTICES

In this, section we consider meet hyperlattices. We define complete meet hyperlattice, and we study some properties of them. Also, we consider some well-known hyperlattices, such as Nakano superlattice and P-hyperlattice, and we show that under certain conditions, they are complete. Then, we define completion of a meet hyperlattice.

Let $(L,\vee,\wedge)$ be a meet hyperlattice. We say L is a strong meet hyperlattice if for all $x, y \in L, y \in x \wedge y$ implies that $x = x \vee y$. Notice that by [26], there exists an order relation on a meet hyperlattice $(L,\vee,\wedge)$ such that $x \leq y$ if and only if $x = x \wedge y$. We say that 0 is a zero element of L, if for all $x \in L$ we have $0 \leq x$ and 1 is a unit element of L if for all $x \in L, x \leq 1$. We say L is bounded if l has 0 and 1. And y is a complement of x if $0 \in x \wedge y$ and $1 = x \vee y$. A complemented hyperlattice is a bounded hyperlattice which every element has a complement. we say L is distributive if for all $x, y, z \in L$.

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. And L is S-distributive if

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Notice that in lattices, the concepts of distributive meet hyperlattice is called a Boolean hyperlattice. Let $\leq$ be an order relation on L such that if $y \leq x$ and for every $a \in L$ implies that $a \wedge y \leq a \wedge x$, then $(L,\vee,\wedge)$ is an order hyperlattice and $\leq$ is a hyper order . Let I and F be nonempty subsets of l. Then I is called an ideal of L if: (1) for every $x, y \in I, x \wedge y \subseteq I$; (2)$x \geq I$ implies $x \in I$.

Also, F is called a filter of L if : (1) for every $x, y \in F, x \vee y \in$ F ; (2) $F \geq x$ implies $x \in F$. *An ideal I of L is* prime if for all $a, b \in L, a \vee b \in I$

Implies that $a \in I$ or $b \in I$. *Similarly*, a filter F is prime if $(a \wedge b) \cup F = \phi$ implies that $a \in F$ or $b \in F$.Let $(L,\vee \wedge)$ be a bounded strong hyperlattice. Then, $p \in L$ is called a prime element if $p = 1$ and when $p = x \vee y$ implies $p = x$ or $p = y$. Let L_1 and L_2 be two meet hyperlattices. The map $f: L_1 \rightarrow L_2$ is called a homomorphism if for all $x, y \epsilon L_1$ *we have* $f(x \wedge y) = f(x) \wedge f(y)$ *and* $f(x \vee y) = f(x) \vee f(y)$. Moreover, f is an isomorphism if it is bijection too.

Let $(L, \vee, \wedge)$ be a meet strong hyperlattices and $L \subseteq A$. The union of all ideals of A is containing L is the ideal generated by A and it is denoted by ⟨A⟩. By [13], in order hyperlattice, we have

$(A] = \{t \in L | \exists\ a \in A, t \geq a\}$.

Theorem 3.1

Let (L, ∨, ∧) be a meet order hyperlattice and X⊆ L. Then we have $\langle X \rangle = (X] \cap (L \wedge X] = \{t \in L |\ t\ \leq x_1 \wedge x_2 \ldots \wedge x_n,\ x_i \in X\}$.

Proof:

First, we have to show that, ⟨X⟩ is an ideal of (L,∨, ∧,≥). Let $t_1,\ t_2 \in \langle X \rangle$, there exists $x_i, y_i \in X$ such that $t_1 \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ *and* $t_2 \geq y_1 \wedge y_2 \wedge \ldots \wedge y_n$. Since ≥ is a hyper order. $t_1 \wedge t_2 \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n \wedge y_1 \wedge y_2 \wedge \ldots \wedge y_n$. Thus $t_1 \wedge t_2 \in \langle X \rangle$. If $x^{'} \geq y$ *and* $y \in \langle X \rangle$, by transitivity of ≥ we have $x^{'} \in X$. Thus, ⟨X⟩ is an ideal. We can easily prove that ⟨X⟩ is the least ideal containing X and the proof is completed.■

Proposition 3.2

Let (L, ∨, ∧) be a meet strong hyperlattice. If for every x, y ∈L, x∧y is an ideal of L, Then x = y.

Proof:

Suppose that x, y ∈L Such that x ∧y is an ideal of L. By previous proposition, there exist $x_1, y_1 \in x \wedge y$ *such that* $x \geq x_1$ *and* $y \geq y_1$. Since x ∧y is an ideal of L and L is strong, we have x = y. ■

Let L be a meet hyperlattice. We can define relation ≽as follows: a ≽ b if and only if b∈ $a \wedge b$. If L is strong , then this relation coincide with ≥ *which is define already*.

Proposition 3.3

Let (L, ∨, ∧) be a distributive meet strong hyperlattice and a ∈L. Then, $I = (a] = \{x \in L | x \geq a\}$ is an ideal of L.

Proof:

Suppose that x, y ∈L such that x ≥ y *and* $y\ \in I = (a]$. Since $x \geq y \geq a$, *we have* $x\ \in I$. Moreover, if p, q ∈ I by distributivity of L, we have $a \vee (p \wedge q) = (a \vee p) \wedge (a \vee q)$. Thus, for every x ∈ $p \wedge q$, *threre exists* y ∈ $p \wedge q$ such that $x = a \vee y \geq a$. Therefore $p \wedge q \supseteq I$. ■

The ideal (a] in proposition 3.4 is called a principal ideal and we denote it by ↓ a.

Definition 3.4

Let (L, ∨, ∧) be a meet hyperlattice. Then, L is complete meet hyperlattice if for every S⊆ L and subsets $S^u = \{x \in L | (\forall s \in S) s = s \vee x\}, S^l = \{x \in L | (\forall s \in S) s \in s \wedge x\}$. S^u *has a greatest element and* S^l has a least element with the order relation ≥ on L.

Note

Consider modular lattice (L, ∨,∧). We recall the Nakano hyper operations ⊔ *and* ⊓ on L. For all x, y ∈L. We define $x \sqcup y = \{z: z \vee x = z \vee y = x \vee y\}, x \sqcap y = \{z \wedge x = z \wedge y = x \wedge y\}$. (L,⊔ ,∧) *is a s*trong join hyperlattice. Similarly, (L, ∨,⊓) is a strong meet hyperlattice.

Theorem 3.5

Let (L, ∨,⊓) be a strong meet hyperlattice which is defined as above and (L,∨, ⋀) be a complete lattice. If the hyperlattice L has greatest and least elements. Then (L,∨,⊓) is a complete hyperlattice.

Proof:

Let L $\subseteq S$. We have $S^l = \{x \in L | \forall s \epsilon S, s \in s \sqcap x\}$. According to the definition of $\sqcap$ *we have* $S^l = \{x \in L | \forall s \in S. s = s \wedge s = s \wedge x\}$. If we denote partial order on a lattice L by $\geq_1$, Then we have $S^l = \{x \in L | \forall s \in S, x \geq_1 s\}$. Since (L,∨,∧) is a complete lattice, so S^l with order $\geq_1$ has a least element x'. Hence, for every s $\in S$ we have, $x' \wedge s = s \wedge s = s \wedge x$. Therefore, $x' \in s \sqcap x'. Thus, s \geq x'$ and similarly we show that S^u has the greatest element with order $\geq$ on L. Now, since lattice L is complete the rest of proof is completed. (Notice that if $S = \emptyset$, then the hyperlattice L should has the greatest and least elements).■

In [15] Konstantinidou investigated P-hyperlattice and defined the hyper operation $\wedge^P$ as follows:

$$a \wedge^P b = a \wedge b \wedge p = \{a \wedge b \wedge p | p \in P\}.$$

She showed that, (L,∨, $\wedge^P$) is a meet strong hyper lattice if and only if for each x∈ L there exists p∈P such that p≥ x.

Theorem 3.6

Let (L,∨,∧) be a complete lattice. Then, the P-hyperlattice (L,∨, $\wedge^P$) is a complete meet hyperlattice.

Proof:

If L⊆ S. Then $S^l = \{x \in L | s \in S, s \in s \wedge^p x\}$. So, there exists p ∈ P such that s = s∧ $x \wedge p$. Therefore, $x \geq_1 s$ where $\geq_1$is order relation on L. Since L is complete, S^l has the lowest element with order relation $\geq_1$. Let x' be such element. Thus for all s∈ S, we have $x' \geq_1 s$. Therefore s∧ $x' = s$. By the condition of L, there exists p∈ P such that p$\geq_1$ s. So, s ∈ s $\wedge^p x'$. Existence of lowest element of S^l is concluded from the complete lattice L. The same arguments can be applied for S^u.■

Definition 3.7

Let L and L' be two meet hyperlattices and $\phi: L' \hookrightarrow L$ be an embedding (onto homeomorphism between meet hyperlattice). If L is a complete meet hyperlattice, then L is a completion of L'.

Theorem 3.8

Let L be a meet strong hyperlattice and $\bar{L} = \{L \subseteq A | A^{ul} = A\}$. Then $\bar{L}$ is a complete meet hyperlattice and $\bar{L}$ is a completion of L.

Proof:

We define $A_1 \wedge A_2 = \cup\{L \subseteq B | B \subseteq A_1 \cap A_2\}$ and $A_1 \vee A_2 = A_1 \cup A_2$. Now, we show that $\bar{L}$ is a meet hyperlattice. We can easily show that $A \in A \wedge A$ and $A_1 \wedge A_2 = A_2 \wedge A_1$. Also, we obtain $A \in A \wedge (A \cup B) = \cup\{C | A \cap (A \cup B) \supseteq C\}$. If A∈ $A \wedge B$ then A⊇ $A \wedge B \supseteq A$. So, $A \vee B = B$. Therefore, ($\bar{L}$,∨,∧) is a meet strong hyperlattice. Now, suppose that$\{A_i\}_{i \in I} = S$ is a family of nonempty subsets of L. Hence, we have $S^l = \{A_j \in L \middle| \forall A_i \in S, A_i \in A_i \wedge A_j\}$. Therefore $\cup A_i$ is the least element of S^l. Similarly we can prove that S^u has the greatest element. Now, we show that $\bar{L}$ is a completion of L. It is enough to define $\phi: L \hookrightarrow \bar{L}$ such that $\phi(x) = x^l$.■

Definition 3.9

Let (L, ∨, ⋀) be a complete meet hyperlattice which is bounded. In addition, the infinite distributive law holds in L, which means that for every $a \in L$ *and* $L \subseteq S$ we have a∨ $\wedge S = \bigwedge_{s \in S}(a \vee s)$. Then, we say L is a D-hyperlattice.

Example 3 Every complete Boolean hyperlattice is a D-hyperlattice.

Proposition 3.10

Let (L, ∨, ∧) be a complete meet strong hyperlattice which has the property that every two elements of L are comparable with the order relation ≥ on L. Then, L is a D-hyperlattice.

Proof:

Let ≥ be an order relation on hyperlattice L. We consider two cases.

i) $a > \wedge S$. In this case, there exists s ∈S such that $s \ngeq a$. Thus, by hypothesis $a \succcurlyeq s$, which means that $s \in s \wedge a$. Since $a \in a \wedge a, a \in s \wedge a$ and L is a strong hyperlattice. Thus, we have $a = s \vee a \; and \; \wedge_{s \in S}(a \vee s) = a. Also, since \; a > S$, we have $a = a \vee s \geq a \vee \wedge S \geq a$. Therefore, $a \vee (\wedge S)= a$.

ii) $\wedge S \geq a$. In this case, $\wedge S \vee a = \wedge S$. And for every s∈ S we have $s \geq a$. Therefore $s = s \vee a$ and $\underset{s \in S}{\wedge}(a \vee s) = \wedge S$. Thus, L is a D-hyperlattice.■

Definition 3.11

Let L be a D-hyperlattice and F be a filter of L. we say F is completely prime if for every $L \subseteq S \; we \; have \; \wedge S \in F \Leftrightarrow s \cup F \neq \emptyset$.

Notice that every completely prime filter in a D-hyperlattice is prime.

Theorem 3.12

Let L be a complete bounded distributive meet strong hyperlattice which is S-distributive. Then $p \in L$ is a prime element if and only if $A = L - \uparrow p$ is a completely prime filter of L.

Proof:

Suppose that p is a prime element. We consider $A = L - \uparrow p$. Then for every $x, y \in A$. We have $x \notin \uparrow p \; and \; y \notin \uparrow p$. If $x \vee y \geq p$, then $x \vee y \in \uparrow p$. Since L is strong $x \vee y \succcurlyeq p, hence \; p \in (x \vee y) \wedge p = (x \wedge p) \vee (y \wedge p)$. Thus, there exists $a \in x \wedge p \; and \; b \in y \wedge p \; such \; that \; p = a \vee b$. Since p is a prime element, we have $p = a \in x \wedge p \; or \; p = b \in y \wedge p$. Thus $x \succcurlyeq p \; or \; y \succcurlyeq p. Therefore, x \geq p \; or \; y \geq p$, and this is a contradiction. Thus $x \vee y \in A. If \; x \geq y \; and \; x \notin p, then \; y \notin p$. Therefore, L is a filter. Now, let $L \subseteq X \; and \wedge X \in A. Hence \; , \wedge X \ngeq p$. Thus, there exists $x \in X \; such \; that \; x \ngeq p$. Therefore $A \cup X \neq \emptyset$. Similarly, we can prove the converse.■

Definition 3.13:

Let L and M be two strong bounded meet hyperlattice and C be a good hyperlattice $(0 \wedge 0 = 0)$. The map $f: L \longrightarrow M$ is a weak homeomorphism if

1)$f(a \wedge b) = f(a) \wedge f(b)$;

2)$f(1) = 1$;

3)$a \vee b = 1 \Leftrightarrow f(a) \vee f(b) = 1$;

4)$x \in a \wedge b \Leftrightarrow f(x) \in f(a \wedge b)$;

Notice that if L is a D-hyperlattice, then the condition (1) can be reformulated as

$f(\wedge S) = \wedge f(S)$ for any $L \subseteq S$. The map f is a homomorphism if for every $L \subseteq S, we \; have \; f(\wedge S) = \wedge f(S) \; and \; f(\vee S) = \vee f(S)$ for every finite subset $L \subseteq S$.

Theorem 3.14

Let $(L,\vee,\wedge)$ be a strong D-hyperlattice which is S-distributive and for every $a,b \in L$ there exists $u,v \in L$ *such that* $u \vee v = 0$ *and* $a \geq u \wedge b$, b≥ $a \wedge v$. Then every weak homomorphism is a homomorphism.

Proof:

By hypothesis, there exists u and v such that $a \geq (u \wedge a)$*and* $b \geq a \wedge v$. Since $a \geq (u \wedge a) \vee (u \wedge b) = u \wedge (a \vee b)$ and $b \geq v \wedge (a \vee b)$. Thus, we have $f(a) \geq f(u) \wedge f(a \vee b)$ and $f(b) \geq f(v) \wedge f(a \vee b)$. Since $f(u) \vee f(v) = 0$, *we have*

$f(a) \vee f(b) \geq f(a \vee b) \wedge \big(f(u) \vee f(v)\big) \geq f(a \vee b) \wedge \big(f(u) \vee f(v)\big) = 0 \wedge f(a \vee b) = f(0) \wedge f(a \vee b)$ as, before we define $a \geq X$ if for every $x \in X, a \geq x$. Thus, since $0 \geq f(a \vee b)$ *and* L is strong,$0 \geq f(a \vee b)$. Therefore, $f(a \vee b) \in 0 = f(0) \wedge f(a \vee b)$. Consequently,

$f(a) \vee f(b) \geq f(a \vee b)$. Since $a \vee b \geq a$ *and* $a \vee b \geq b$, *we have* $f(a \vee b) \geq f(a) \vee f(b)$. Thus the proof is completed and f is a homomorphism.■

Definition 3.16

Let $(L,\vee,\bigvee)$ be a distributive meet hyperlattice. We say that L is a regular hyperlattice if for every $a,b \in L$ with the condition $a \ngeq b$, there exists $y \in L$ such that $0 \in a \wedge y$ *and* $a \vee y = 1$.

Example 4

Every complemented distributive hyperlattice is a regular hyperlattice.

Theorem 3.17

Let $(L,\vee,\wedge)$ be a regular meet hyperlattice such that for every $x \in L$, $x \wedge 0 = x$. Then, every weak homomorphism in the category of meet hyperlattice is a homomorphism.

Proof:

Suppose that $a,b \in L$. If $a \geq b$ *or* $b \geq a$. Then the proof is trivial. So, we consider $a \ngeq b$ *and* $b \ngeq a$. Therefore, by the regularity of L, there exists $y, y' \in L$ *such that* $0 \in a \wedge y, a \vee y = 1$ and $0 \in b \wedge y'$, $y' \vee b = 1$. Thus, $0 \in (y \wedge y') \wedge (a \vee b)$ and we have

$$f(a) \vee f(b) = \big(f(a) \vee f(b)\big) \vee 1 \supseteq \big(f(a) \vee f(b)\big) \vee \big(f(y) \wedge f(y') \wedge f(a \vee b)\big) = 1 \wedge 1 \wedge (f(a) \vee f(b) \vee f(a \vee b)) \geq f(a \vee b).$$

The reverse inclusion is trivial. Hence, the proof is completed.■

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