0 – DISTRIBUTIVE MEET – SEMILATTICE

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Abstract: *In this paper, we have studied some properties of ideals and filters of a meet-semilattice. We have discussed 0-distributive meet-semilattice and given several characterizations of 0-distributive meet-semilattices directed below. Finally, we have included a generalization of prime separation theorem in terms of dual annihilators.*

Keywords: ideals, meet-semilattice,0-distributive lattice, dual annihilator.

1. Introduction:

Varlet [7] have given the definition of a 0-sistributive lattice. Then Balasubramani et al [1] have established some results on this topic. A lattice L with 0 is called a 0-distributive lattice if for all a, b, c $\in L$ with a $\wedge b = 0 = a \wedge c$ imply a $\wedge (b \vee c) = 0$. Any distributive lattice with 0 is 0 – distributive. In this paper we will study the 0-distributive meet-semilattices.

An ordered set (S, $\leq$) is said to be a meet – semilattice if inf{a, b} exists for all a, b$\in S$.we write a $\wedge b$ in place of inf{a, b}.

A meet- semilattice s is called distributive if $a \geq b_1 \wedge b_2 (a, b_1, b_2 \in S)$

Implies the existence of $a_1, a_2 \in S$; $a_1 \geq b_1$, $a_2 \geq b_2$ with $a = a_1 \wedge a_2$.

For literature on meet- semilattice, we refer the reader to consult Talukder et al [5,6], Noor et l [3] and Gratzer [2].

A meet-semilattice S with 0 is said to be 0-distributive if for any $a, b, c \in S\ such\ that\ a \wedge b = 0 = a \wedge c$ implies that $a \wedge d = 0$ for some $d \geq b, c$.

Both distributive and modular meet-semilattices share a common property "For all $a, b \in S$ there exists $c \in S\ such\ that\ c \geq a, b$".this property is known as the directed below property. Hence a meet-semilattice with this property is known as a directed below semilattice.

A subset I of a meet-semilattice S is called an upset if $x \in L\ and\ y \in S$ with $x \geq y\ implies\ y \in L$.

Let S be a meet- semilattice. A non-empty subset F of S is called a filter if

(1) F is an upset, and

(2) $a, b \in F$implies there exists $d \geq a, b\ such\ that\ d \in F$.

A filter F is called proper filter of a meet- semilattice S if $F \neq S$.

A proper filter (upset) F in S is called a prime filter (upset) if$a \wedge b \in F$

Implies either $a \in F\ or\ b \in F$. For $a \in S$, the filter

F = $\{ x \in S | x \geq a\}$ is called the principle filter generated by [a). A prime upset (filter) is called a maximal prime upset (filter) if it does not contain any other prime upset (filter).

A subset I of S is called an ideal if

(1) $a, b \in I\ implies\ a \wedge b \in I (tt)\ a \in S, t \in I\ with\ a \geq t\ implies$

$a \in I$.

An ideal I of a meet-semilattice S is called prime ideal if $I \neq S$ and $s - I$ is a prime filter.

A minimal ideal I of S is a proper ideal which is not contained in any other proper ideal. That is, if there is a proper ideal J such that $J \subseteq I$ $then$ $i = J$.

Let S be a meet-semilattice with 0. For $S \subseteq A$.

Set $A^{\perp d} = \{x \in S | x \wedge a = 0 \ \forall \ a \in A\}$.Then $A^{\perp d}$ is called the dual annihilator of A. This is always an upset but not necessarily a filter.

For $a \in S$, we $denote$

$\{a\}^{\perp d}$=$\{x \in S | x \wedge a = 0\}$. $moreover$ $A^{\perp d} = \bigcup_{a \in A}\{\{a\}^{\perp d}\}$.

2. Some properties if ideals and filters of a meet-semilattice

Lemma.2.1

Let S be a meet semilattice with 0. Then every prime upset contains a maximal prime upset.

Proof:

Let F be a prime upset of S and let A denote the set of all prime upset Q contained in F. Then A is non empty as $F \in A$. Let C be a chain in A and

Let $M = \cup \{X | X \in C\}$. We claim that M is a prime upset. M is non empty as $0 \in M$.Let $a \in M$ and $a \geq b$.Then $a \in X$ for all $X \in C$. Hence $b \in X$ for all $X \in C$ as X is an $upset$.

Thus $b \in M$. Again let$x \wedge y \in M$ for $some$ $x, y \in S$.Then $x \wedge y \in X$ for all $X \in C$.Since X is prime upset ,so either $x \in X$ or $y \in X$ $this$ $implies$ $either$ $x \in M$ or $y \in M$. Hence M is a prime upset . Therefore, we can apply to A the dual form of Zorn's lemma to conclude the existence of a maximal member of A.■

Lemma.2.2

Let S be a directed below meet-semilattice. Then the union of any two filters of S is also a filter.

Proof:

Let F, Q be two filters of a directed below meet-semilattice S. Let $a \in F \cup Q$ and $b \in S$ $with$ $b \geq a$. Then $a \in F$ and $a \in Q$. Since both F and Q are filters. So $b \in F$ and $b \in Q$. Hence $b \in F \cup Q$. Again let $a, b \in F \cup Q$. So $a, b \in F$ and $a, b \in Q$. Since F and Q are both filters, then there exists $f \in F$ and $q \in Q$ $such$ $that$ $f, q \geq a, b$. Let $C = f \vee q$. Then $c \in F \cup Q$, $where$ $C \geq a, b$. $hence$ $F \cup Q$ is a $filter$.■

Lemma.2.3

Let I be a nonempty proper subset of a meet-semilattice S. Then I is an ideal if and only if S-I is a prime upset.

Proof:

Let I be an ideal of a meet-semilattice S. Now let $x \in S - I$ and

$x \geq y$, $then$ $x \notin I$, so $y \notin I$ as I is an $ideal$. Hence $y \in S - I$, $thus$ $S - I$ is an $upset$.Since I is an ideal, so $S - I$is an upset. Since I is an ideal, so $S - I \neq S$, $therefore$ $S - I$ is a $proper$ upset. Let $a, b \in S$ $with$ $a \wedge b \in S - I$. $then$ $a \wedge b \notin I$. Therefore either $a \notin I$ or $b \notin I$ as I is an ideal. Hence either $a \in S - I$ or $b \in S - I$. Therefore $S - I$ is a prime upset.

Conversely let S-I is a prime upset and $x, y \in I$, $then$ $x, y \notin S - I$.

Thus $x \wedge y \notin S - I$ as S-I is a prime upset. Hence $x \wedge y \in I$.Again , let $x \in I \; and \; y \geq x. then \; x \notin S - I, therefore \; y \notin S - I \; as \; S - I \; is \; an \; upset. Hence \; y \in I \; and \; thus \; I \; is \; an \; ideal.$ ■

Corollary.2.4

Let I be a nonempty subset of a meet-semilattice S. Then I is a minimal ideal if and only if S-I is a maximal prime upset.

Theorem.2.5

Every proper ideal of a meet-semilattice S with 0 is contained in a minimal ideal.

Proof:

Let I be a proper ideal in S with 0. Let P be the set of all proper ideals containing I. then P is nonempty as $I \in P$. Let C be a chain in P and let $M = \cap \{X | X \in C\}$. We claim that M is an ideal with $M \subseteq I$. Let $x \in M \; and \; y \geq x$. Then $x \in X \; for \; some \; X \in C$. Hence $Y \in X$ as X is an ideal. Therefore $y \in M$.Again , Let $x, y \in M, then \; x \in X \; and \; y \in Y \; for \; some \; x, y \in C$. Since C is a chain, so either $Y \subseteq X \; or \; X \subseteq Y$. Suppose $Y \subseteq X, so \; x, y \in Y, then \; x \wedge y \in Y$ as Y is an ideal. Hence $x \wedge y \in M$, moreover I contain M, so M is minimal element of C. then by Zorn's lemma, P maximal element say Q with $Q \subseteq I$.■

Now we give a characterization of minimal ideals of a meet-semilattice.

Theorem.2.6

Let S be a meet-semilattice with 0. A proper ideal M in S is minimal if and only if for any element $a \in S - M$. there exists an element $b \in M$ such that $a \wedge b = 0$.

Proof:

Suppose M is minimal and $a \notin M, let \; a \wedge b \neq 0 \; for \; all \; b \in M$. Consider $M_1 = \{y \in S | y \geq a \wedge b \; for \; some \; b \in M\}$. Clearly M_1 is an ideal and is proper as $0 \notin M_1 \; for \; every \; b \in M$. We have $b \geq a \wedge b$ and so $b \in M_1. Thus \; M_1 \subseteq M$. Also $a \notin M \; but \; a \in M_1$, so $M_1 \subset M$,

Which contradicts the minimality of M. Hence there must exists some $b \in M \; such \; that \; a \wedge b = 0$.

Conversely, if the proper ideal M is not minimal, then as $0 \in S$, there exists a minimal ideal N such that $N \subset M$, for any element $a \in N - M$. There exists an element $b \in M \; such \; that \; a \wedge b = 0$. Hence $a, b \in N$ imply $0 = a \wedge b \in N$ which is contradiction. Thus, M must be a minimal ideal.■

3. SOME CHARACTERIZATIONS OF 0 – DISTRIBUTIVE MEET SEMILATTICE

In this section, we prove our main results of this paper.

Theorem 3.1

Every 0 – distributive meet semilattice is directed below.

Proof:

Let S be a 0 – distributive meet semilattice and b, c $\in S$. Then a∧b= 0 = $0 \wedge c$ which implies there exists d$\in S$ with d $\geq b, c$ such that a $\wedge \, d = 0$. Thus d is upper bound of b, c. The converse of the above theorem is not true by s_2 of figure 1.1. ■

Theorem 3.2

Let a, $a_1, a_2, \dots, a_n$ be elements of a 0 – distributive meet semilattice S such that a $\wedge \, a_1 = a \wedge a_2 = \cdots = a \wedge a_n = 0$. Then a $\wedge \, b = 0$ for some b $\geq a_1, a_2, \dots, a_n$.

Proof:

We want to prove this theorem using mathematical induction method.

Let $a \wedge a_1 = a \wedge a_2 = 0$. Since S is 0 – distributive. So, a $\wedge b_1 = 0$ for some $b_1 \geq a_1, a_2$. That is, the statement is true for $a_1 and \ a_2$. Let , $a \wedge a_1 = a \wedge a_2 = \cdots = a \wedge a_{k-1} = 0$.Then for the 0 – distributivity of S, $a \wedge b_2 = 0$ for some $b_2 \geq b_2, a_k$ as S is a 0 – distributive. This implies that a $\wedge b = 0$ for some b $\geq a_{1,} a_2, \dots, a_k$. Hence by the method of mathematical induction, the theorem is true for b $\geq a_{1,} a_2, \dots, a_n$.■

Following results gives some nice characterization of 0 – distributive meet semilattices.

Theorem 3.3

For a directed below meet semilattice S with 0, the following conditions are equivalent: i) S is a 0 – distributive. ii) $\{a\}^{\perp d}$ is a filter for all a ∈ S. iii) $A^{\perp d}$ is a filter for all finite subsets A of S. iv) Every minimal idea is prime.

Proof:

i)⇔ii):

Let $x \in \{a\}^{\perp d}$ and x ≥ y. Since $x \in \{a\}^{\perp d}$, so we get a ∧ x = 0 implies a ∧ y = 0 as x ≥ y. Hence $y \in \{a\}^{\perp d}$,and so $\{a\}^{\perp d}$is an upset. Again let $x, y \in \{a\}^{\perp d}$, Thus a$\wedge x = a \wedge y = 0$. By 0 – distributivity of S, there exists z with z $\geq x, y$ such that a$\wedge z = 0$. Therefore z ∈ $\{a\}^{\perp d}$, and so $\{a\}^{\perp d}$ is a filter.

Conversely, let x, y, z ∈ S with x$\wedge y = x \wedge z = 0$. Then y, z ∈ $\{x\}^{\perp d}$. Since $\{x\}^{\perp d}$ is a filter, so there exists t ≥ y, z such that t ∈ $\{x\}^{\perp d}$, and so t$\wedge x = 0$.This implies S is 0 – distributive.

ii) ⇔iii):

It is trivial by theorem 2.2 as $A^{\perp d} = \bigcup_{a \in A} \{a\}^{\perp d}$.

i)⇒ iv):

Let I be a minimal ideal of S. Then by corollary 2.4, $S - I$ is a maximal prime upset. Now suppose x, y ∈ $S - I$. Then x, y∉ I, and so by the minimality of I, $I \wedge (x] = S, I \wedge (y] = S$. This implies a$\wedge x = 0 = b \wedge y$ for some a, b ∈ I. Thus a$\wedge b \wedge x = a \wedge b \wedge y = 0$. Since S is 0 – distributive , there exists d≥ x, y such that a $\wedge b \wedge d = 0$.

Now, a $\wedge b \in I \ implies \ a \wedge b \ \notin S - I$, and $S - I$ is prime implies d∈ $S - I$. Therefore $S - I$ is a prime filter and so I is a prime ideal.

iv)⇒i):

Let S be not 0 – distributive. Then there are a, b, c ∈ S such that $a \wedge b = 0 = a \wedge c \ and \ a \wedge d \neq 0 \ for \ all \ d \geq b, c$. Now, set $I = \{x \in S | x \geq a \wedge y, y \geq b, c\}$. Clearly I is an ideal and it proper as 0∉ I.By theorem2.5 $I \subseteq J$ for some minimal ideal J. Now we claim that either b∈ $J \ or \ c \in J$. If b, c∉ $J, then \ b, c \ \notin S - J$. As J is a prime ideal, then we have S−J is a prime filter and b, c ∈ $S - J$. Since S−J is a filter, there is c ∈ $S - J \ such \ that \ a \leq b, c$. Hence $a \wedge e \in S - J$ gives a contradiction. Hence b∈ $J \ or \ c \in J$. This implies, either $a \wedge b \in J \ or \ a \wedge c \in J$. Thus 0∈ J which contradict the minimality of J. Therefore$a \wedge d = 0 for \ some \ d \leq b, c \ and$ hence S is a 0-distributive.■

Note that in case of a 0-distributive lattice L, for any A⊆ $L, A^{\perp d}$ is a filter. But this is not true in a directed below meet-semi lattice S with 0, as the union of finite number of filters in S is not necessarily a filter.

Corollary 3.4

In a 0-distributive meet-semi lattice, every proper ideal is contained in a prime ideal.

This immediately follows by theorem 2.5 and theorem 3.3.

Theorem 3.5

In a 0-distributive meet-semi lattice S if {0}≠ A is the union of all filters of S not equal to {0}. Then $A^{\perp d} = \left\{x \in S \middle| \{x\}^{\perp d} \neq \{0\}\right\}$.

Proof:

Let x∈ $A^{\perp d}$. Since x∧ $a = 0$ $for\ all\ a \in A$.Since A≠ {0}, so $\{x\}^{\perp d} \neq \{0\}$. Thus x ∈ $R.H.S$. That is $A^{\perp d} \subseteq R.H.S$.

Conversely, let x∈ $R.H.S$. So $\{x\}^{\perp d} \neq \{1\}$.Also S is a 0-distributive. Then $\{x\}^{\perp d}$ is a filter of S. Hence A⊆ $\{x\}^{\perp d}$and so $A^{\perp d} \subseteq \{x\}^{\perp d}$. This implies x∈ $A^{\perp d}$. Thus R.H.S. ⊆ $A^{\perp d}$ which completes the proof.■

Finally, we give a necessary and sufficient condition for a meet-semi lattice S with 0, 0 be a 0-distributive which is a generalization of power and et al. [4; Theorem 7].

Theorem 3.6

Let S be a meet-semi lattice with 0. Then S is 0-distributive if and only if for any ideal I disjoint with $\{x\}^{\perp d}$ $(x \in S)$, there exists a prime ideal containing I and disjoint with $\{x\}^{\perp d}$.

Proof:

Suppose S is a 0-distributive meet-semi lattice. Let P be the set of has I∈ P. Let C be a chain in P and let $M = \cap\{X|x \in C\}$. First we claim that M is an ideal with M⊆ $I\ and\ M \cup \{x\}^{\perp d} - \emptyset$. Let x ∈ $M\ and\ x \leq y$. Then x∈ $X\ for\ some\ X \in C$. Hence y∈ $X\ as\ X$is an ideal. Thus y∈ M. Again, le t x, y∈ M. Then x ∈ $X\ and\ y \in Y\ for\ some\ X, Y \in C$. Since C is a chain, so either X⊆ $Y\ or\ Y \subseteq X$. Suppose X⊆ $Y, so\ x, y \in Y$. Then $x \wedge y \in Y$ as Y is an ideal. Hence $x \wedge y \in M$. Thus M is an ideal. Moreover, I contain M and M∪ $\{x\}^{\perp d} = \emptyset$.Then by Zorn$'s$ lemma, there exists a minimal element Qin P. Hence by Zorn's lemma as in theorem 2.5, there exists a minimal ideal I containing P and disjoint from $\{x\}^{\perp d}$. We claim that x∈ P. If not, then P∧ $(x]$ is an ideal containing P. By the minimality of P, (P∧ $(x]$) ∪ $\{x\}^{\perp d} = \emptyset$. Let t ∈ (P∧ $(x]$) ∪ $\{x\}^{\perp d}$. Then t≤ $p \wedge x$ for some p∈ $P\ and\ t \wedge x = 0$. This implies that $p \wedge x = 0\ and\ so\ p \in \{x\}^{\perp d}$, which is a contradiction. Now suppose y∉ $P. Then\ (P \wedge [y]) \cap \{x\}^{\perp d} \neq \emptyset\ bythe\ minimality\ of\ P$. Let S∈ $(P \wedge (y]) \cup \{x\}^{\perp d}$. Then S≤ $p_1 \wedge y$ for some $p_1 \in P\ and\ S \wedge x = 0$. This implies $(p_1 \wedge x) \wedge y = 0. Since\ p_1 \wedge x \in P$, so by theorem 2.6, P is a minimal ideal of S. Therefore, by theorem 3.3, P is a prime ideal.

Conversely, let x, y, z∈ S such that $x \wedge y = 0, x \wedge z = 0$. Suppose for all y, z≤ d we have $x \wedge d \neq 0$. Then d∉ $\{x\}^{\perp d}$. Set I = $\{a \in S | a \leq x \wedge a, for\ alla \leq y, z\}$. First we claim that I is a proper ideal. Clearly, I is nonempty as $x \in I. Let\ p \in I\ and\ p \leq q$. Then $p \leq x \wedge a\ and\ so\ q \leq x \wedge a$. Thus $p \wedge q \leq x \wedge a$. Hence $p \wedge q \in I$. Therefore I is an ideal and I is a proper ideal as a∉ I. Again $x \in I\ and\ a \in I\ for\ all\ a \leq y, z$. Then $\{x\}^{\perp d} \cup I = \emptyset$ and hence there is a prime ideal J such that J⊆ $I\ and\ \{x\}^{\perp d} \cup J = \emptyset$. Thus $x \in J\ and\ a \in J\ for\ all\ a \leq y, z$. Now we claim that either y∈ $J\ or\ z \in J$. If y, z ∉ $J\ then\ y$, z ∈ $S - J$. As J is a prime ideal, then $S - J$ is a prime filter and y, z∈ $S - J$. Since $S - J$ is a filter, there is f∈ $S - J$ such that f≤ y, z which is a contradiction. Hence either $y \in J\ or\ z \in J$. This implies either $x \wedge y \in J\ or\ x \wedge z \in J$. Thus 0∈ J which is a contradicts the primeness of J. Hence $x \wedge d = 0$. Thus S is a 0-distributive.■

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