DIRECT PRODUCT OF SOFT HYPER LATTICES

Gayathri.V[1], Gayathri.G [2]

[1,2](Masters in Mathematics), Thassim Beevi Abdul kader College for women, Kilakarai, Ramanathapuram (dist).

---***---

Abstract: *In this article we noticed about the direct product of soft hyper lattices. Also, we give some definition and theorem related to the poset.*

Key words: Hyper lattice, Poset, Direct product, modular soft hyper lattice, Distributive soft hyper lattice.

I. Introduction:

In this chapter, we introduce the concept of direct product of soft hyper lattice and we prove that the direct product of any two distributive soft hyper lattice is a soft lattice and the direct product of any two modular soft hyper lattices is a modular soft lattice.

II. Preliminaries:

2.1. Definition:

Let $L \subseteq S(U)$ and $\vee$ and $\wedge$ be two binary operations on L. is equipped with two commutative and associative binary operations $\vee$ and $\wedge$ which are connected by absorption law, then algebraic structure $(L, \vee, \wedge)$ is called soft lattice.

2.2. Definition:

A distributive lattice is a lattice in which $\vee$ and $\wedge$ distributive over each other in for all x, y, z in lattice the distributivity laws are satisfied:

1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

2.3. Definition:

A modular lattice is any lattice which satisfies the modular law.

$$M: x \leq y \rightarrow x \vee (y \wedge z) \approx y \wedge (x \vee z)$$

The modular law is obviously equivalent to the identity.

$$(x \wedge y) \vee (y \wedge z) \approx y \wedge ((x \wedge y) \vee z)$$

Since $a \leq b$ holds iff $a = a \wedge b$. Also, it is not difficult to see that every lattice satisfies

$$x \leq y \rightarrow x \vee (y \wedge z) \leq y \wedge (x \vee z)$$

So, to verify the modular law it suffices to check implication,

$$x \leq y \rightarrow y \wedge (x \vee z) \leq x \vee (y \wedge z)$$

2.4. Definition:

A relation R on a set S is called partial order

- Reflexive

- Antisymmetric
- Transitive

A set S together with a partial ordering R is called poset. Using notation,

a < b when (a, b) ∈ R

a ≮ b when (a, b) ∈ R, a ≠ b

2.5. Definition:

(L, ∨, ∧) is a hyper lattice. For all a, b, c ∈ L

1. a ∈ a ∨ a, a ∧ a = a
2. a ∨ b = b ∨ a, a ∧ b = b ∧ a
3. (a ∨ b) ∨ c = a ∨ (b ∨ c)
   (a ∧ b) ∧ c = a ∧ (b ∧ c)
4. a ∈ [a∧ (a ∨ b)] ⋂ [a∨ (a ∧ b)]
5. a ∈ a ∨ b ⇒ a ∧ b = b

III. Main Result:

3.1. Theorem

Let (R, ∨, ∧) and (S, ∨, ∧) be two soft hyper lattices then (R×S, ∨, ∧) is a soft hyper lattice.

Proof:

We have to prove that binary operations ∨ and ∧ defined on R×S satisfies.

1. a ∈ a ∨ a, a ∧ a = a
2. a ∨ b = b ∨ a, a ∧ b = b ∧ a
3. (a ∨ b) ∨ c = a ∨ (b ∨ c)
   (a ∧ b) ∧ c = a ∧ (b ∧ c)
4. a ∈ [a∧ (a ∨ b)] ⋂ [a∨ (a ∧ b)]
5. a ∈ a ∨ b ⇒ a ∧ b = b

let, (gx_1, gy_1), (gx_2, gy_2) and $(gx_3, gy_3) \in R×S$

then,

i. $(gx_1, gy_1) \vee (gx_1, gy_1) = (gx_1 \vee gx_1)\ \ (gy_1 \vee gy_1)$

$= (gx_1, gy_1)$

Similarly, we get

$(gx_1, gy_1) \wedge (gx_1, gy_1) = (gx_1, gy_1)$

ii. $(gx_1, gy_1) \vee (gx_2, gy_2) = (gx_1 \vee gx_2, gy_1 \vee gy_2)$

$= (gx_2 \vee gx_1, gy_2 \vee gy_1)$

$= (gx_2, gy_2) \vee (gx_1, gy_1)$

Similarly, we get

$(gx_1, gy_1) \wedge (gx_2, gy_2) = (gx_2, gy_2) \wedge (gx_1, gy_1)$

iii. $((gx_1, gy_1) \vee (gx_2, gy_2)) \vee (gx_3, gy_3) = (gx_1, gy_1) \vee (gx_2 \vee gx_3, gy_2 \vee gy_3)$

$= (gx_1 \vee (gx_2 \vee gx_3), gy_1 \vee (gy_2 \vee gy_3))$

$= ((gx_1 \vee gx_2) \vee gx_3, (gy_1 \vee gy_2) \vee gy_3)$

$= (gx_1 \vee gx_2), (gy_1 \vee gy_2) \vee (gx_3, gy_3)$

$= (gx_1, gy_1) \vee (gx_2, gy_2) \vee (gx_3, gy_3)$

Similarly, we get

$((gx_1, gy_1) \wedge (gx_2, gy_2)) \wedge (gx_3, gy_3) = (gx_1, gy_1) \wedge (gx_2, gy_2) \wedge (gx_3, gy_3)$

iv. $(gx_1, gy_1) \vee ((gx_1, gy_1) \wedge (gx_2, gy_2)) = (gx_1, gy_1) \vee (gx_1 \wedge gx_2, gy_1 \wedge gy_2)$

$= (gx_1 \vee (gx_1 \wedge gx_2), gy_1 \vee (gy_1 \wedge gy_2))$

$= (gx_1, gy_1)$

Similarly, we get

$(gx_1, gy_1) \wedge ((gx_1, gy_1) \vee (gx_2, gy_2)) = (gx_1, gy_1)$

v. $(gx_1, gy_1) \in [(gx_1, gy_1) \vee (gx_2, gy_2)] = (gx_1, gy_1) \wedge (gx_2, gy_2)$

$= (gx_2, gy_2)$

Thus (R×S, ∨, ∧) is a soft hyper lattice.

3.2. Theorem:

Let $(R, \leq_1)$ and $(S, \leq_2)$ be two soft hyper lattices. Define a relation ≤ on R×S as follows: for

(gx_1, gy_1) and $(gx_2, gy_2) \in R$S $(gx_1, gy_1) \leq (gx_2, gy_2)$ if and only if $gx_1 \leq_1 gx_2$ and

$gy_1 \leq_2 gy_2$. Then (R×S, ≤) is a soft hyper lattice.

Proof:

First, we claim that ≤ is a partial order on R×S.

Let $(R, \leq_1)$ and $(S, \leq_2)$ be two soft hyper lattices, (gx_1, gy_1), (gx_2, gy_2) and $(gx_3, gy_3) \in R$S. Then $gx_1, gx_2, gx_3 \in R$ and $gy_1, gy_2, gy_3 \in S$.

i. Since $gx_1 \leq_1 gx_1$ and $gy_1 \leq_2 gy_2$ we get $(gx_1, gy_1) \leq (gx_1, gy_1)$.

Therefore, ≤ is reflexive.

ii. Suppose $(gx_1, gy_1) \leq (gx_2, gy_2)$ and $(gx_2, gy_2) \leq (gx_1, gy_1)$.

Then $gx_1 \leq_1 gx_2$ and $gy_1 \leq_2 gy_1$. Also, $gx_2 \leq_1 gx_1$ and $gy_2 \leq_2 gy_1$.

Since $gx_1, gx_2 \in R$, $gx_1 \leq_1 gx_2$ and $gx_2 \leq_1 gx_1$, we get $gx_1 = gx_2$.

Since $gy_1, gy_2 \in S$, $gy_1 \leq_2 gy_2$ and $gy_2 \leq_2 gy_1$, we get $gy_1 = gy_2$.

Thus $(gx_1, gy_1) = (gx_2, gy_2)$.

Hence ≤ is anti-symmetric.

iii. Suppose $(gx_1, gy_1) \leq (gx_2, gy_2)$ and $(gx_2, gy_2) \leq (gx_3, gy_3)$.

Then $gx_1 \leq_1 gx_2$ and $gy_1 \leq_2 gy_2$. Also, $gx_2 \leq_1 gx_3$ and $gy_2 \leq_2 gy_3$.

Since $gx_1, gx_2, gx_3 \in R$, $gx_1 \leq_1 gx_2$ and $gx_2 \leq_1 gx_3$, we get , $gx_1 \leq_1 gx_3$.

Since $gy_1, gy_2, gy_3 \in S$, $gy_1 \leq_2 gy_2$ and $gy_2 \leq_2 gy_3$, we get $gy_1 \leq_2 gy_3$.

Thus $(gx_1, gy_1) \leq (gx_3, gy_3)$.

Hence ≤ is transitive.

Therefore, (R×S, ≤) is a poset.

Now, we have to prove that l.u.b $\{(gx_1, gy_1), (gx_2, gy_2)\}$ and g.l.b$\{(gx_1, gy_1), (gx_2, gy_2)\}$

exist in R×S.

Since R and S are soft hyper lattices, $gx_1 \leq gx_1 \vee gx_2$ and $gy_1 \leq gy_1 \vee gy_2$.

Therefore, $(gx_1, gy_1) \leq (gx_1 \vee gx_2, gy_1 \vee gy_2)$

$= (gx_1, gy_1) \vee (gx_2, gy_2)$.

Similarly, $(gx_2, gy_2) \leq (gx_1 \vee gx_2, gy_1 \vee gy_2)$

$= (gx_1, gy_1) \vee (gx_2, gy_2)$.

Thus $(gx_1, gy_1) \vee (gx_2, gy_2)$ is an upper bound of $\{(gx_1, gy_1), (gx_2, gy_2)\}$.

Suppose (gx_3, gy_3) is an upper bound for $\{(gx_1, gy_1), (gx_2, gy_2)\}$.

Then $(gx_1, gy_1) \leq (gx_3, gy_3)$ and $(gx_2, gy_2) \leq (gx_3, gy_3)$.

Therefore, $gx_1 \leq gx_3$ and $gy_1 \leq gy_3$. Also, $gx_2 \leq gx_3$ and $gy_2 \leq gy_3$.

That is, $gx_1 \leq gx_3$ and $gx_2 \leq gx_3$, and $gy_1 \leq gy_3$ and $gy_2 \leq gy_3$.

Since R and S are soft hyper lattices, $gx_1 \vee gx_2 \leq gx_3$ and $gy_1 \vee gy_2 \leq gy_3$.

That is, $(gx_1 \vee gx_2, gy_1 \vee gy_2) \leq (gx_3, gy_3)$.

Thus $(gx_1, gy_1) \vee (gx_2, gy_2) \leq (gx_3, gy_3)$.

Hence $(gx_1, gy_1) \vee (gx_2, gy_2)$ is the least upper bound of $\{(gx_1, gy_1), (gx_2, gy_2)\}$.

Similarly, we can prove that $(gx_1, gy_1) \wedge (gx_2, gy_2)$ is the greatest lower bound of

$\{(gx_1, gy_1), (gx_2, gy_2)\}$.

Thus l.u.b $\{(gx_1, gy_1), (gx_2, gy_2)\}$ and g.l.b $\{(gx_1, gy_1), (gx_2, gy_2)\}$ exist for every

(gx_1, gy_1) and (gx_2, gy_2) in R×S.

Hence (R×S, ≤) is a soft hyper lattice. $(gx_1, gy_1) \vee (gx_2, gy_2)$.

IV. Direct Product of distributive and Modular Soft hyper Lattices

4.1. Theorem:

Two soft hyper lattices R and S are distributive if and only if R□ is a distributive soft hyper lattice.

Proof:

Let R and S be two distributive soft hyper lattices. Let (gx_1, gy_1), (gx_2, gy_2) and $(gx_3, gy_3) \in$ R□. Then $gx_1, gx_2, gx_3 \in$ R and $gy_1, gy_2, gy_3 \in$ S.

Since R and S are distributive soft hyper lattices, we get

$gx_1 \vee (gx_2 \wedge gx_3) = (gx_1 \vee gx_2) \wedge (gx_1 \vee gx_3)$ and

$gy_1 \vee (gy_2 \wedge gy_3) = (gy_1 \vee gy_2) \wedge (gy_1 \vee gy_3)$.

Now $(gx_1, gy_1) \vee ((gx_2, gy_2) \wedge (gx_3, gy_3))$

$= (gx_1, gy_1) \vee (gx_2 \wedge gx_3, gy_2 \wedge gy_3)$

$= (gx_1 \vee (gx_2 \wedge gx_3), gy_1 \vee (gy_2 \wedge gy_3))$

$= ((gx_1 \vee gx_2) \wedge (gx_1 \vee gx_3), (gy_1 \vee gy_2) \wedge (gy_1 \vee gy_3))$

$= (gx_1 \vee gx_2, gy_1 \vee gy_2) \wedge (gx_1 \vee gx_3, gy_1 \vee gy_3)$

$= ((gx_1, gy_1) \vee (gx_2, gy_2)) \wedge ((gx_1, gy_1) \vee (gx_3, gy_3))$

Therefore, R□ is a distributive soft hyper lattice.

Conversely, let R□ be a distributive soft hyper lattice.

We have to prove that both R and S are distributive soft hyper lattices.

Let $gx_1, gx_2, gx_3 \in$ R and $gy_1, gy_2, gy_3 \in$ S.

Then (gx_1, gy_1), (gx_2, gy_2) and $(gx_3, gy_3) \in$ R□. Since R□ is a distributive soft hyper lattice,

$(gx_1, gy_1) \vee (gx_2, gy_2) \wedge (gx_3, gy_3) = ((gx_1, gy_1) \vee (gx_2, gy_2)) \wedge ((gx_1, gy_1) \vee (gx_3, gy_3))$

$\Longrightarrow (gx_1, gy_1) \vee (gx_2 \wedge gx_3, gy_2 \wedge gy_3) = (gx_1 \vee gx_2, gy_1 \vee gy_2) \wedge (gx_1 \vee gx_3, gy_1 \vee gy_3)$

$\Longrightarrow (gx_1 \vee (gx_2 \wedge gx_3), gy_1 \vee (gy_2 \wedge gy_3))$

$= ((gx_1 \vee gx_2) \wedge (gx_1 \vee gx_3), (gy_1 \vee gy_2) \wedge (gy_1 \vee gy_3))$

$\Longrightarrow gx_1 \vee (gx_2 \wedge gx_3) = (gx_1 \vee gx_2) \wedge (gx_1 \vee gx_3)$ and

$gy_1 \vee (gy_2 \wedge gy_3) = (gy_1 \vee gy_2) \wedge (gy_1 \vee gy_3)$.

Thus, R and S are distributive soft hyper lattices.

4.2. Theorem:

Two soft hyper lattices R and S are modular if and only if R⊠ is a modular soft hyper lattice.

Proof:

Let R and S be two modular soft hyper lattices. Let (gx_1, gy_1), (gx_2, gy_2) and $(gx_3, gy_3) \in$ R⊠.

Suppose $(gx_1, gy_1) \leq (gx_3, gy_3)$.

Then $gx_1, gx_2, gx_3 \in R$. Since R is a modular soft hyper lattice and $gx_1 \leq gx_3$,

$$gx_1 \vee (gx_2 \wedge gx_3) = (gx_1 \vee gx_2) \wedge gx_3.$$

Also, $gy_1, gy_2, gy_3 \in S$. Since M is a modular soft hyper lattice and $gy_1 \leq gy_3$,

$$gy_1 \vee (gy_2 \wedge gy_3) = (gy_1 \vee gy_2) \wedge gy_3.$$

Now $(gx_1, gy_1) \vee ((gx_2, gy_2) \wedge (gx_3, gy_3))$

$$= (gx_1, gy_1) \vee (gx_2 \wedge gx_3, gy_2 \wedge gy_3)$$

$$= (gx_1 \vee (gx_2 \wedge gx_3), gy_1 \vee (gy_2 \wedge gy_3))$$

$$= ((gx_1 \vee gx_2) \wedge gx_3, (gy_1 \vee gy_2) \wedge gy_3)$$

$$= (gx_1 \vee gx_2, gy_1 \vee gy_2) \wedge (gx_3, gy_3)$$

$$= ((gx_1, gy_1) \vee (gx_2, gy_2)) \wedge (gx_3, gy_3).$$

Therefore, R⊠ is a modular soft hyper lattice.

Conversely,

Let R⊠ be a modular soft hyper lattice.

We have to prove that both R and S are modular soft hyper lattices.

Let $gx_1, gx_2, gx_3 \in R$ with $gx_1 \leq gx_3$ and $gy_1, gy_2, gy_3 \in S$ with $gy_1 \leq gy_3$.

Since R⊠ is a modular soft hyper lattice, we have

$$(gx_1, gy_1) \vee ((gx_2, gy_2) \wedge (gx_3, gy_3)) = ((gx_1, gy_1) \vee (gx_2, gy_2)) \wedge (gx_3, gy_3).$$

$$\Rightarrow (gx_1, gy_1) \vee (gx_2 \wedge gx_3, gy_2 \wedge gy_3) = (gx_1 \vee gx_2, gy_1 \vee gy_2) \wedge (gx_3, gy_3)$$

$$\Rightarrow (gx_1 \vee (gx_2 \wedge gx_3), gy_1 \vee (gy_2 \wedge gy_3)) = ((gx_1 \vee gx_2) \wedge gx_3, (gy_1 \vee gy_2) \wedge gy_3)$$

$$\Rightarrow gx_1 \vee (gx_2 \wedge gx_3) = (gx_1 \vee gx_2) \wedge gx_3 \text{ and } gy_1 \vee (gy_2 \wedge gy_3) = (gy_1 \vee gy_2) \wedge gy_3.$$

Thus, R and S are modular soft hyper lattices.

References

[1] G. Birkhoff, Lattice Theory, First edition, Colloquium Publications, vol. 25, Amer. Math. Soc., Providence, R. I., 1940.

[2] R. P. Dilworth, Structure and Decomposition Theory, Proceedings of Symposia on Pure Mathematics: Lattice Theory, vol. II, Amer. Math. Soc., Providence, R. I., 1961.

[3] X.Z. Guo and X. L. Xin, Hyperlattice, Pure Appl. Math., 20 (2004), 40-43.

[4] S.W. Han and B. Zhao, Distributive hyperlattice, J. Northwest Univ., 35(2) (2005),125-129.

[5] E.K.R. Nagarajan and P. Geetha, Direct product of soft lattices, Pub. in the pro. of Inter. Conf. on Rec. Trend. in Discrete Math. and its Appl. to Sci. and Eng. Dec 3rd (2013), 32-42.

[6] E.K.R. Nagarajan and P. Geetha, Soft Atoms and Soft Complements of Soft Lattices, International Journal of Science and Research, 3(5) (2014), 529-534.

[7] R. P. Dilworth, Proof of a conjecture on finite modular lattices, Ann. of Math. 60 (1954), 359–364.