A Note on Lie Derived Length of Group Algebras

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Abstract

Let KG be the group algebra of a group G over a field K of characteristic p. In this paper we have focused on Lie derived length of group algebras.

1 Introduction

Let KG be the group algebra of a group G over a field K of characteristic $p > 0$. If we introduce an operation $[x, y] = xy - yx$, where $x, y \in KG$. Then KG becomes a Lie algebra w.r.t + and [ ], which is said to be associated Lie algebra of KG and is denoted by $L(KG)$. If M and N are any two Lie ideals of KG, then $[M, N]$ is also a Lie ideal of KG generated by $\{[m, n] | m \in M, n \in N\}$. Let $x_1, x_2, x_3, ..., x_n \in KG$, then the left normed Lie commutators are defined by $[x_1, x_2] = x_1x_2 - x_2x_1$ and, inductively, $[x_1, x_2, ..., x_n] = [[x_1, x_2, ..., x_{n-1}], x_n]$. A Lie derived series of KG is defined inductively as follows

$$KG = \delta^{[0]}(L(KG)) \supseteq \delta^{[1]}(L(KG)) \supseteq \delta^{[2]}(L(KG))... \supseteq \delta^{[m]}(L(KG)) \supseteq ...,$$

where

$$\delta^{[0]}(L(KG)) = KG \text{ and for } n \geq 0$$

$$\delta^{[n+1]}(L(KG)) = [\delta^{[n]}(L(KG)), \delta^{[n]}(L(KG))]$$

KG is Lie solvable if $\delta^{[n]}(L(KG)) = \{0\}$ for some $n \in N$ and the least non negative integer n such that $\delta^{[n]}(L(KG)) = \{0\}$ is called the Lie derived length of KG and it is denoted by $dl_L(KG)$.

A strong Lie derived series of KG is defined inductively as follows:

$$KG = \delta^{(0)}(KG) \supseteq \delta^{(1)}(KG) \supseteq \delta^{(2)}(KG)... \supseteq \delta^{(m)}(KG) \supseteq ...,$$

where

$$\delta^{(0)}(KG) = KG \text{ and for } n \geq 0$$

$$\delta^{(n+1)}(KG) = [\delta^{(n)}(KG), \delta^{(n)}(KG)]KG$$

KG is strongly Lie solvable if $\delta^{(n)}(KG) = 0$ for some $n \in N$ and the least non negative integer n such that $\delta^{(n)}(KG) = 0$ is called the strong Lie derived length of KG and it is denoted by $dl^L(KG)$.

Passi, Passman and Sehgal [1] proved that the group algebra KG is Lie solvable if and only if one of the following conditions holds:

1. G is abelian;
2. Char$K = p > 0$ and G' is a finite $p-$ group;
3. Char$K = 2$ and G contains a subgroup L of index 2, such that L' is a finite $2-$group.

It is also well known that KG is strongly Lie solvable if and only if one of the following conditions holds:

1. G is abelian;
2. Char$K = p > 0$ and G' is a finite $p-$ group.

Thus we can say that for odd characteristic KG is Lie solvable if and only if it is strongly Lie solvable. However in general, $dl_L(KG) \neq dl^L(KG)$. Calculation of the derived length of group algebras is more difficult.After the characterization of Lie solvable group algebras by Passi et al.the first significant result was classification of Lie metabelian group algebras, due to Levin and Rosenberger. There are other authors also like Shalev, Balogh, Juhasz, Spinelli who have done great work in this area.

Definition. *If Char$K = p$, where p is a prime number, and G contains an element of order p, then KG is called Modular Group Algebra.*

Definition. *A group G is called an Engel group, if for each pair $(x, y) \in G \times G$ $\exists$ a non negative integer $n = n(x, y)$ such that $(x,\ _ny) = 1$.*

Definition. *An Engel group is called n-Engel group, if $(x,\ _ny) = 1\ \forall\ x, y\ \in G$ i.e. every element of G is both a left and a right n-Engel element.*

2 Results

Let KG be the group algebra. It is easy to see that the group algebras KG has derived length 1 if and only if G is abelain. Levin and Rosenberger [2] characterized Lie metabelian group algebras i.e. group algebras of derived length 2. They proved the following theorem

Theorem 2.1. *Let KG be the group algebra of a group G over a field K of characteristic p. Then KG is Lie metabelian if and only if one of the following conditions holds:*

- *$p = 3$ and G' is central of order 3;*

- *$p = 2$ and G' is central and elementary abelian $2-$group of order atmost 4.*

In the same paper, it is also proved that $dl^L(KG) = 2$ if and only if $dl_L(KG) = 2$.

Definition. *A group algebra KG is said to be Lie centrally metabelian if ,*

$$\delta^{[2]}(L(KG)) \subseteq Z(KG)$$

or equivalently
$[[x_1, x_2], [x_3, x_4], x_5] = 0$ *for all* $x_1, x_2, x_3, x_4, x_5 \in KG$.

The classification of Lie centrally metabelian group algebras was started by Sharma and Srivastava [3]. In 1992, they proved that if Char$K = p > 3$, then KG is Lie centrally metabelian if and only if G is abelian. Also they proved that if Char$K = 3$ and KG is Lie centrally metabelian, then commutator subgroup of G is a finite 3-Engel 3-group of exponent atmost 9 and consequently, nilpotent of class atmost 4. In 1996, Kulshammer and Sharma [4] completed the above characterization for $p = 3$ and proved that KG is Lie centrally metabelian if and only if $|G'| \leq 3$. On the other hand Sahai and Srivastava [5] also classified Lie centrally metabelian group algebras whenever characteristic of the field is 3. Thus Lie centrally metabelian group algebras for odd characteristic were completely characterized. In view of these articles a non commutative group algebra KG of characteristic $p > 2$ is Lie centrally metabelian if and only if $p = 3$ and $G' \cong C_3$. Lie centrally metabelian group algebras for even characteristic were classified by Rossmanith [6, 7]. He proved the following result

Theorem 2.2. *Let KG be the group algebra of a group G over a field K of characteristic $p = 2$. Then KG is Lie centrally metabelian, if and only if one of the following conditions holds:*

- *$|G'|$ divides 4;*
- *G' is central and elementary abelian of order 8;*
- *G acts by element inversion on $G' \cong Z_2 \times Z_4$, and $C_G(G')' \subseteq \Phi(G')$;*
- *G contains an abelian subgroup of index 2.*

Working in the same direction Sahai [8] classified strongly Lie solvable group algebras of strong Lie derived length 3. She proved that for a group G and for a field K of characteristic $p \neq 2$, $\delta^{(3)}(KG) = 0$ if and only if one of the following conditions holds:

- G is abelian;
- $p = 7$, $G' = C_7$ and $\gamma_3(G) = 1$,

- $p = 5$, $G' = C_5$ and either $\gamma_3(G) = 1$ or $\gamma_n(G) = G'$ for all $n \geq 3$ with $y^g = y^{-1}$ for all $y \in G'$ and for all $g \notin C_G(G')$;
- $p = 3$ and G' is a group of one of the following types:
  1. $G' = C_3$;
  2. $p = 5$, $G' = C_3 \times C_3$ and either $\gamma_3(G) = 1$ or $\gamma_3(G) = C_3$, $\gamma_4(G) = 1$ or $\gamma_n(G) = G'$ for all $n \geq 3$ with $y^g = y^{-1}$ for all $y \in G'$ and for all $g \notin C_G(G')$;
  3. $G' \cong C_3 \times C_3 \times C_3, \gamma_3(G) = 1$.

It is also shown that whenever $p \geq 7$, then following are equivalent

1. $\delta^{(3)}(KG) = \{0\}$
2. $\delta^{[3]}(L(KG)) = \{0\}$.

In 2010, a classification of Lie solvable group algebras of derived length 3 over fields of characteristic 3 and 5 is given by Sahai [9]. The author has proved that

1. If G is an arbitrary non abelian group and $Char\ K = 5$, then $\delta^{[3]}(L(KG)) = \{0\}$ if and only if $G' \cong C_5$ and $x^g = x^{-1}$ for all $x \in G'$ and for all $g \notin C_G(G')$.
2. If G is a 2-Engel group and $Char\ K = 3$, then $\delta^{[3]}(L(KG)) = \{0\}$ if and only if G' is central and elementary abelian 3-subgroup of G s.t. $|G'| \leq 3^3$.
3. If $Char\ K = 5$ and G is an arbitrary group, then $\delta^{(3)}(KG) = \{0\}$ if and only if $\delta^{[3]}(L(KG)) = \{0\}$.
4. If $Char\ K = 3$ and G is a 2-Engel group, then $\delta^{(3)}(KG) = \{0\}$ if and only if $\delta^{[3]}(L(KG)) = \{0\}$.

Working in the same direction, Chandra and Sahai [10] focused on group algebras in characteristic 3 of those groups which are not $2-$Engel but have abelian commutator subgroup.The main theorem is that

Theorem 2.3. *For a group G with abelian commutator subgroup G', $\delta^{[3]}(L(KG)) = \{0\}$ iff one of the following conditions holds:*

- $G' = C_3$;
- $G' = C_3 \times C_3$ *and* $y^g = y^{-1}$ *for all* $y \in G'$, $g \notin C_G(G')$.

It is also shown that if G' is abelian and Char$K = 3$, then $\delta^{[3]}(L(KG)) = \{0\}$ is equivalent to $\delta^{(3)}(KG) = \{0\}$. Also they proved that if G is a non abelian torsion group having no element of order 2 such that $G'' = 1$ and $\delta^{[3]}(L(KG)) = \{0\}$, then G is nilpotent of class atmost 3.

After that Siwach, Sahai and Srivastava [11] classified strongly Lie solvable group algebras of derived length 4.

Theorem 2.4. *Let G be a group and K be a field of characteristic $p > 5$. Then $\delta^{(4)}(KG) = \{0\}$, if and only if one of the following conditions holds:*

- *G is abelian;*
- *$p = 13$, $G' = C_{13}$ and $\gamma_3(G) = 1$;*
- *$p = 11$, $G' = C_{11}$ and $\gamma_3(G) = 1$;*
- *$p = 7$, G' is a group of one of the following types*
  1. *$G' = C_7$;*
  2. *$G' = C_7 \times C_7$ and $\gamma_3(G) = 1$,*

Thus it is observed that Lie solvable group algebras are receiving a lot of attention. We have seen that a lot of work has been done on Lie derived length and strong Lie derived length of group algebras. But still there are much more to do. There are still a lot of open problems in this area which are yet to be solved.

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