GENERALIZED DERIVATIONS IN LIE ALGEBRA

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Abstract - We prove that the Lie Product [f1, f2] ∈ GD(L), the set of all generalized derivations on an arbitrary Lie algebra L over a fixed field K. We also show that inner derivations fa form an Ideal in derivation Algebra D(L). To this result, Jacobson [2] proposition 2 page 10 comes out as a corollary. Also we show that f1f2 may be the generalized derivation iff f1 and f2 have some possibilities.

Key Words: Lie Algebra, Generalized derivations, Inner derivation, Ideal, Jacobi identity.

INTRODUCTION

We use the notation and terminology of JACOBSON [2] unless stated otherwise Havala [1] gave the following definition "Let R be a ring. The additive map f: R → R will be called a generalized derivation if ∃ a derivation d of R s.t. f(xy) = f(x)y + x d(y) ∀x,y ∈ R"

Let L be an arbitrary Lie algebra over a field K and GD(L) be the set of all generalized derivations on L then in section 1, we prove that Lie product [f1,f2] ∈ GD(L) ∀ f1, f2 ∈ GD(L). In Proposition 1.2, we show that inner derivations fa form an ideal in derivation algebra D(L) when L is the Lie algebra. To this result, Jacobson [2] proposition 2 page 10 comes out as a corollary.

In Theorem 1.4 we have proved that if GD(L) is not closed with respect to multiplication of generalized derivations but in Theorem 2.1, it is our attempt to prove that f1f2 may be generalized derivation iff f1 and f2 have some possibilities. If f1(x) = λx+μf2(x) where λ,μ ∈R and f2 is an arbitrary generalized derivation. Then f1 is also a generalized derivation.

1. Generalized Differential Algbera

Let L be an arbitrary Lie algebra over a field K. A generalized derivation f in L is a linear mapping of L onto L satisfying

f(xy) = f(x)y + xD(y) ∀x,y ∈L where D is the derivation in L.

Let GD(L) be the set of all generalized derivation of L. If f1,f2 ∈GD(L) then

(f1 + f2)(xy) = f1(xy) + f2(xy) = f1(x)y + xD1(y) + f2(x)y + xD2(y)

⇒ (f1 + f2)(xy) = (f1 + f2)(x)y + x(D1 + D2)(y) Hence f1 + f2 ∈GD(L).

It is easy to show that αfi ∈GD(L) ∀ α ∈ k, fi ∈GD(L). Now

(f1f2)(xy) = f1(f2(x)y + xD2(y)) = f1(f2(x)y) + f2(x)D1(y) + f1(x)D2(y) +xD1(D2(y))

Interchanging of 1, 2 and subtraction we get

[f1,f2](xy) = ([f1,f2](x))(y) + x([D1,D2](y)) where [f1,f2] = f1f2 −f2f1 and [D1,D2] = D1D2 −D2D1. Hence [f1,f2] ∈GD(L). So GD(L) is a subalgebra of L where L is the algebra of linear transformation in the vector space L. We call this, generalized differential algebra of L.

Theorem 1.1 (Jacobi Identity)

If f1,f2,f3 ∈GD(L) then

[[f1,f2],f3] + [[f2,f3],f1] + [[f3,f1],f2] = 0 ∀ f1,f2,f3 ∈GD(L)

Proof: We have proved if f1,f2 ∈GD(L)

Then [f1,f2] = f1f2 −f2f1) ∀ GD(L)

Now

[[f1,f2],f3] = [f1,f2]f3 −f3[f1,f2]

= (f1f2 −f2f1)f3 −f3(f1f2 −f2f1)

= f1f2f3 −f2f1f3 −f3f1f2 + f3f2f1

Similarly

[[f2,f3],f1] = f2f3f1 −f3f2f1 −f1f2f3 + f1f3f2 [[f3,f1],f2]

= f3f1f2 −f1f3f2 −f2f3f1 + f2f1f3

Adding, we get

[[f1,f2],f3] + [[f2,f3],f1] + [[f3,f1],f2] = 0

∀f1,f2,f3 ∈GD(L)

1.2

we also get proposition 2 of Jacobson [2] page 10 "If L " is Lie algebra, then the inner derivations form an ideal J(L) in the derivation algebra D(L) where D(L) is the set of all derivations in L which becomes a sub algebra of L where L is the algebra of linear transformations in the vector space L. (By virtue of D1 + D2 ∈ D(L), [D1,D2] ∈ D(L), Di being the derivations.)

Remarks: Let L be the lie algebra, $a \in L$ Then

$fa(xy) = fa(x)y + xDa(y) = (xa-ax)y + x(ya-ay)$

$= xay-axy + xya-xay$

$\Rightarrow fa(xy) = xya-axy \ \forall \ x,y \in L$

$= (xy)a-a(xy)$

We call fa, inner derivation by a.

Now

$fa = aR - aL$

$\Rightarrow fa(xy) = (aR - aL)(xy)$

$= (xy)a-a(xy)$

We get the following

(1) $[DaR] = (D(a))R$ $[Dfa] = fDa$

(2) $(Dfa - faD)(xy) = Dfa(xy)-fa(D(xy))$

$= D(xya-axy)-fa(D(x)y + xD(y))$

$= xyD(a)-D(a)xy = fD(a)(xy)$

Hence from above calculation we get

Proposition 1.3

If L is Lie algebra then the inner derivation fa form an Ideal J(L) in the derivation algebra D(L).

Proof: Now

$[fa,D] = faD-Dfa$

$\Rightarrow [fa,D](xy) = (faD-Dfa)(xy)$

$= faD(xy)-Dfa(xy)$

$= fa(D(x)y + xD(y))-D(xya-axy)$

$= fa(D(x)y) + fa(xD(y))-D(xya) + D(axy)$

$= D(x)ya-aD(x)y + xD(y)a-axD(y)$

$-D(xy)a - xyD(a) + D(a)xy + aD(xy)$

$= D(x)ya-aD(x)y + xD(y)a-axD(y)$

$-D(x)ya - xD(y)a-xyD(a) + D(a)xy$

$+ aD(x)y + axD(y)$

$= D(a)xy-xyD(a)$

$= -(xyD(a)-D(a)xy)$

$= -fD(a)$

$\Rightarrow [fa,D] \in J(L)$

$\Rightarrow$ J(L) is an ideal Hence the theorem

Corollary 1.3.1

Replacing f by D we get Jacobson [2] proposition 2 page 10.

Theorem 1.4

If $f1, f2 \in GD(L)$ then $f1f2 \notin GD(L)$

Proof: Now

$f1f2(xy) = f1(f2(xy))$

$= f1(f2(x)y + xD2(y))$

$\Rightarrow f1f2(xy) = f1(f2(xy)) + f1(xD2(y))$

$= f1(f2(x)y + f2(x)D1(y) + f1(x)D2(y)$

$+ xD1(D2(y)))$

$= f1(f2(x))y + f2(x)D1(y) + f1(x)D2(y)$

$+ xD1(D2(y))$

$= f1f2(x)y + f2(x)D1(y) + f1(x)D2(y)$

$+ xD1D2(y)$

$\Rightarrow f1f2(xy) \neq f1f2(x)y + xD1D2(y)$

Hence $f1f2 \notin GD(L)$

Hence proved.

Remark: We can say that GD(L) is not closed with respect to the multiplication of generalized derivations

2. It is our attempt in this part that f1f2 may be generalized derivation iff f1 and f2 have some possibilities. Also if f1 depends on f2 where f2 be an arbitrary generalized derivation as in Theorem 2.1 given below. Then f1 is also a generalized derivation.

Theorem 2.1

If f2 is an arbitrary generalized derivation and $f1(x) = \lambda x+\mu f2(x)$ where $\lambda, \mu \in R$. Then f1 is generalized derivation.

Proof: Now

$f1(xy) = \lambda xy + \mu f2(xy)$

$= \lambda xy + \mu(f2(x)y + xD2(y))$

= λxy + μf2(x)y + μxD2(y)

= (λx + μf2(x))y + μxD2(y)

= f1(x)y + x(μD2(y))

We take μD2(y) = D1(y)

⇒ f1(xy) = f1(x)y + xD1(y)

⇒ f1 is generalized derivation. Hence proved

3. CONCLUSION

In this Paper, we proved the results "Lie product [f1,f2] ∈GD(L) ∀ f1, f2 ∈GD(L) and inner derivations fa form an ideal in derivation algebra D(L) when L is the Lie algebra." Then by the virtue of these results, Jacobson [2] proposition 2 page 10 comes out as a corollary. It is also proved that f1f2 may be generalized derivation iff f1 and f2 have some possibilities and if f1(x) = λx + μf2(x) where λ,μ ∈ R and f2 is an arbitrary generalized derivation. Then f1 is also a generalized derivation.

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BIOGRAPHIES

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