On Distributive Meet-Semilattices

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Abstract - Desire by Gentzen’s conjunction elimination ruling the Natural Deduction calculus and reading discrimination with join in a natural move, we perceive a perception of Different notions of distributivity for meet-semilattices. In the move we substantiate that those perceptions are linearly ordered. The conjunction -conditional portion of antagonistic logic.

Keywords: Meet-semilattice, ND-distributive, K-distributive, H-distributive

1 Introduction

Different notions of distributivity for semilattices have been introduced in the article as a ratiocination of the usual distributive property in lattices. As far as we know, notions of distributivity for semilattices have been given. We perceive a notion of distributivity for meet-semilattice. That will be call ND-distributivity. We aspiration to find out whether it is correspondent to any of the perceptions previously present in the article. In doing so, we also analyze the contrasting notion of distributivity for meet-semilattice we have found. Namely, we see that the accustomed perceptions entail each other in the following linear order:

\[ GS \Rightarrow K \Rightarrow (H \Rightarrow LR \iff ND) \Rightarrow B \Rightarrow S_n \Rightarrow S_{n-1} \Rightarrow \cdots \Rightarrow S_3 \Rightarrow S_2 \]

and we also afford countermodels for the reciprocals. Additionally, we show that H-distributivity may be detect as a very natural adaptation of a move to characterize distributivity for lattice, case that will afford more desire for the use of that perceptions. Indicate that Hickman used the term balmy distributivity for H-distributivity.

2 Preliminaries

Definition 2.1

An algebra \( < L, \wedge, \vee > \), is called a lattice if \( L \) is a nonempty set, \( \wedge \) and \( \vee \) are binary operation on \( L \). Both \( \wedge \) and \( \vee \) are idempotent, commutative and associative and they satisfy the absorption law. The study of lattices is called the lattice theory.

Properties 2.2

An algebra \( < L, \wedge, \vee > \), if \( L \) is nonempty set,

i. \( a \vee b = b \vee a \)
ii. \( a \wedge b = b \wedge a \)
iii. \( a \vee (b \vee c) = (a \vee b) \vee c \)
iv. \( a \wedge (b \wedge c) = (a \wedge b) \wedge c \)
v. \( a \vee (a \wedge b) = a \)
vi. \( a \wedge (a \vee b) = a \)

for all \( a, b, c \in L \)

Definition 2.3

A semilattices is a structured \( S = < S, \cdot > \) where \( \cdot \) is a binary operations called the semilattices operation such that

i. is associative \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \)
ii. is commutative \( x \cdot y = y \cdot x \)
iii. is idempotent \( x \cdot x = x \)
Definition 2.4

Distributive join-semilattice are defined dually.

A join-semilattice is distributive if for all $a, b$ and $x$.

If $X \leq a \vee b$ then there exists $a'$ and $b'$ such that $a' \leq a, b' \leq b$ and $X = a' \vee b'$

Definition 2.5

A meet-semilattice is distributive if for all $a, b$ and $X$.

If $a \wedge b \leq X$ then there exists $a'$ and $b'$ such that $a \leq a', b \leq b'$ and $X = a' \wedge b'$

Definition 2.6

In this section we provide the basic notions and notations that will be used in the paper.

Let $J = (J, \leq)$ be a poset. For any $J \subseteq S$, we will use the notations $S^l$ and $S^u$ to denote the set lower and upper bound of $S$, respectively. That is,

$S^l = \{x \in J : s \leq x, \text{ for all } s \in S\}$

$S^u = \{x \in J : x \leq s, \text{ for all } s \in S\}$

Lemma 2.7

Let $J = (J, \leq)$ be a poset. For all $a, b, c \in J$ the following statement are equivalent.

i. For all $x \in J$, if $a \leq x$ and $b \leq x$, then $c \leq x$.

ii. $\{c\}^l \subseteq \{a, b\}^l$

iii. $c \in \{a, b\}^u$

A poset $J = (J, \leq)$ is a meet-semilattice if $\inf \{a, b\}$ exists for every $a, b \in J$. A poset $J = (J, \leq)$ is a lattice it is meet-semilattice. As usual the notations $a \wedge b$ shall stand for $\inf \{a, b\}$.

Given a meet-semilattice $J = (J, \leq)$, we will use the following notions:

- $J$ is upwards directed iff for any $a, b \in J$, there exists $c \in J$ such that $a \leq c$ and $b \leq c$.
- A nonempty subset $I \subseteq J$ is said to be an ideal iff
  1. if $x, y \in I$, then $x \wedge y \in I$ and
  2. if $x \in I$ and $x \leq y$, then $y \in I$.
- The principal ideal generated by an element $a \in A$, noted $[a]$, is defined by $[a] = \{x \in A : a \leq x\}$.
- $\text{Id}(J)$ will denote the set of all ideals of $J$.
- $\text{Id}_{fp}(J)$ will denote the subset of ideals that are union of a finite set of principal ideals, that is, $\text{Id}_{fp}(J) = \{(a_1) \cup ... \cup (a_k) : a_1, ..., a_k \in J\}$

In this paper we are concerned with various notions of distributivity for meet-semilattice, all of them generalizing the usual notion of distributive lattice, that is a lattice $J = (J, \leq)$ is distributive if the following equation holds true for any elements $a, b, c \in J$:

(D) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

There are several equivalent formulation of this property, in particular we mention the following ones that are relevant for this paper:

- For all $a, b, c \in J$, if $a \wedge b = a \wedge c$ and $a \wedge b = a \wedge c$ then $b = c$. 

For any two ideals $I_1, I_2$ of $J$, the ideal $I_1 \land I_2$ generated by their intersection is defined by $I_1 \land I_2 = \{a \land b : a \in I_1, b \in I_2\}$.

The set $\text{Id}(J)$ of ideals of $J$ is a distributive lattice.

For the case of semilattices, several non-equivalent generalizations of these conditions can be found in the article, already mentioned in the introduction. However, as expected, all of them turn to be equivalent to usual distributivity in the case of lattices.

In the contrast, in any sense of distributivity for meet-semilattices that coincides with usual distributivity in the case of a lattice, the class of distributive meet-semilattice is not even a quasi-variety.

Definition 2.8

A meet-semilattice $J = (J, \leq)$ is called ND-distributive (ND for Natural Deduction) if it satisfies $(D_\land)$.

3 Different notions of distributivity for meet-semilattices

3.1 GS-distributivity

The following seems to be the most popular definition of distributivity for meet-semilattices.

Definition 1

A meet-semilattice $J = (J, \leq)$ is GS-distributive iff $(GS)$ for all $a, b, x \in J$, if $a \land b \leq x$, then there exist $a', b' \in J$ such that $a \leq a', b \leq b', and x = a' \land b'$

Proposition 1

Let $J = (J, \leq)$ be a meet-semilattice. Then the following two statements are equivalent:

i. Every pair of elements has upper bound.
ii. For all $a, b, x \in J$ if $a \land b \leq x$, then there exist $a', b' \in J$ such that $a \leq a', b \leq b'$ and $x \leq a' \land b'$

Proof:

(i) $\Rightarrow$ (ii) Suppose $a \land b \leq x$. Let $a'$ be a upper bound of $\{a, x\}$ and $b'$ be a upper bound of $\{b, x\}$. Then $a \leq a'$ and $b \leq b'$. Also, $x \leq a'$ and $x \leq b'$, which implies that $x \leq a' \land b'$

(ii) $\Rightarrow$ (i) Let $a, b \in J$. We have $a \land b \leq a$. Then by hypothesis there exist $a \leq a'$, $b \leq b'$ such that $a \leq a' \land b'$. As $a' \land b' \leq b'$, it follows that $a \leq b'$. Then $a, b \leq b'$ that is $b'$ is a upper bound of $\{a, b\}$

3.2 K-distributivity

The concept given in the following definition is similar to the one in (GS).

Definition 2

A meet-semilattice $J = (J, \leq)$ is K-distributive iff $(K)$ for all $a, b, x \in J$, if $a \land b \leq x$, $a \not\leq x$ and $b \not\leq x$, then there exist $a', b' \in J$ such that $a \leq a', b \leq b'$ and $x = a' \land b'$

Proposition 2

GS-distributivity implies K-distributivity, but not conversely.
Proof

The most simple counter-example showing that the reciprocal does not hold is the meet-semilattice in Figure [1], that is not upward directed. Indeed, the given meet-semilattice is K-distributive, as the only way to satisfy the antecedent of (K) is to take \( a \land b \leq 0 \), but then the consequent is also true. On the other hand, it is not (GS) distributive, as we have \( a \land b \leq a \) and however, there are no \( a \leq a', b \leq b' \) such that \( a' \land b' = a \).

Figure 1: Meet-semilattice showing that K-does not imply GS-distributivity

3.3 H-distributivity

In [10] Hickman introduces the concept of mildly distributive join-semilattices as those join-semilattices whose lattice of their strong ideals is distributive. In [10, Theorem 2.5, p.200] it is stated that it is equivalent in the following statement:

\[(H) \text{ for all } n \text{ and } x, a_1, ..., a_n, \]
\[\text{If for all } b \text{ (if } b \leq a_1, ..., b \leq a_n \text{ then } b \leq x), \]
Then there exists \((x \lor a_1) \land ... \land (x \lor a_n) \text{ and } (x \lor a_1) \land ... \land (x \lor a_n) \leq x\)

The given conditional may be seen as a translation of the following version of distributivity for lattices:

\[\text{If } a_1 \land ... \land a_n \leq x, \text{ Then } (x \lor a_1) \land ... \land (x \lor a_n) \leq x\]

In the case of a meet-semilattice \( J = (J, \leq) \) and using quantifiers, (H) may be rendered as follows:

For all \( n \) and \( x, a_1, ..., a_n \in J \),
\[\text{If } a_1 \land ... \land a_n \leq x, \]
Then for all \( y \), if for all \( i = 1, ..., n \) \((\text{for all } z, \text{if } x \leq z \text{ and } a_i \leq z, \text{then } y \leq z)\)
\[\text{Then } y \leq x\]

That is in turn equivalent to:

For all \( n \) and \( x, a_1, ..., a_n \in J \),
\[\text{If } a_1 \land ... \land a_n \leq x, \]
Then for all \( y \), if \((\text{for all } z, \text{if } x \leq z \text{ and } (a_1 \leq z \text{ or } ... \text{ or } a_n \leq z), \text{Then } y \leq z)\)
\[\text{Then } y \leq x.\]

Using set-theoretic notation, (H) may also be rendered as follows:

\[\text{(C) for all } n \text{ and } x, a_1, ..., a_n \in J, \]
\[\text{If } a_1 \land ... \land a_n \leq x, \text{ then } x \in (\{x, a_1\} \cap ... \cap \{x, a_n\})\]

At this point, the reader may wonder whether the number \( n \) of arguments is relevant or whether two arguments are enough. Let us settle this question. Firstly, with that in mind, consider

\[\text{(D, } n) \text{ for all } x, a_1, ..., a_n, c, \]
\[\text{If } [c]^l \subseteq \{x, a_1\} \cap ... \cap \{x, a_n\}, \text{ then } [c]^l \subseteq \{x, a_1 \land ... \land a_n\}\]

Now, let us state the following fact.
Lemma 2

\((D_{an})\) is equivalent to (C).

**Proof**

\(\Rightarrow\) Suppose \(a_1 \land \ldots \land a_n \leq x\) and \(y \in \{\{x, a_1\} \land \ldots \land \{x, a_n\}\}^u\). Our goal is to see that \(y \leq x\). Take \(c = y\) and apply \((D_{an})\). Then we have \(\{y\}^l \subseteq \{x, a_1 \land \ldots \land a_n\}^l = \{x\}^l\), and hence \(y \leq x\).

\(\Leftarrow\) Suppose \(\{c\}^l \subseteq \{x, a_1\}^l \land \ldots \land \{x, a_n\}^l\). We have to prove that if \(x \leq y\) and \(a_1 \land \ldots \land a_n \leq y\), then \(c \leq y\). Now, using (C) and the assumptions \(x \leq y\) and \(a_1 \land \ldots \land a_n \leq y\) it follows that \(y \in (\{x, a_1\} \land \ldots \land \{x, a_n\})^u \supseteq \{c\}^ul \supseteq \{c\}^l\) hence \(c \leq y\).

Proposition 3

Let \(J = (J, \leq)\) be a meet-semilattice. Then, K-distributivity implies H-distributivity.

**Proof:**

Suppose

\((X1)\) for all \(x \in J\), if \(h \leq x\) and \(a \leq x\), then \(c \leq x\) and

\((X2)\) for all \(x \in J\), if \(h \leq x\) and \(b \leq x\), then \(c \leq x\).

Further, suppose \(h \leq x\) and (S1) \(a \land b \leq x\). The goal is to prove \(c \leq x\). Let us suppose that \(a \leq x\). Then, using (X1) and (S1), it follows that \(c \leq x\). The case \(b \leq x\) is analogous using (X2). Finally, suppose both \(a \leq x\) and \(b \leq x\). Using (K) and (S2) it follows that there exist \(a', b' \in J\) such that \(a \leq a', b \leq b'\) and (F) \(x = a' \land b'\), which implies \(x \leq a', which using (S1) gives \(h \leq a'. As we also have a \leq a', using (X1) we get c \leq a'. Reasoning analogously, we get c \leq b'. So, using (F) it follows that c \leq x."

3.4 LR-distributivity

Larmerova-Rachunek version of distributivity (see [13]) was given for poset, as we next see.

**Definition 3**

A poset \(P = (P, \leq)\) is LR-distributivity iff

\((LRP)\) for all \(a, b, c \in P\)

\[\{(c, a) \land \{c, b\}\}^u = \{(c) \land \{a, b\}\}^l\]

**Remark 1**

In the given definition, it is enough to take one inclusion. Indeed, given a poset \(P = (P, \leq)\) and \(a, b, c \in P\), it is always the case that

\[\{(c) \land \{a, b\}\}^l \subseteq \{(c, a) \land \{c, b\}\}^u\].

**Definition 4**

A meet-semilattice \(J = (J, \leq)\) is LR-distributive iff

\((LR)\) for all \(a, b, c \in J\)

\[\{(c, a) \land \{c, b\}\}^u \subseteq \{c, a \land b\}^l\]

Now, it can be seen that LR-distributivity is equivalent to H-distributivity, and hence to the condition \((D_{an})\).

**Proposition 4**

Let \(J = (J, \leq)\) be a meet-semilattice. Then the following conditions are equivalent:
i. J satisfies (LR)
ii. J satisfies (H)
iii. J satisfies (D₀)

Proof

The equivalence between (ii) and (iii) is a meet-semilattice J = (J; ≤) being H-distributive if and only if it is ND-distributive. Let us prove that (LR) implies (H). Suppose

(X1) for all x ∈ J, if h ≤ x and a ≤ x, then c ≤ x

(X2) for all x ∈ J, if h ≤ x and b ≤ x then c ≤ x, h ≤ x and a ∧ b ≤ x

Then, the last two inequalities imply x ∈ {c, a ∧ b} ↓ . So, using (LR)

we get that x ∈ ((c, a) ↓ ∩ (c, b) ↓)↑, that is, for all y ∈ J, if y ∈ ((c, a) ↓ ∩ (c, b) ↓)↑, then y ≤ x. Now, it should be clear that (X1) and (X2) imply that c ∈ ((c, a) ↓ ∩ (c, b) ↓)↑. So c ≤ x, as desired.

Now, let us see that (H) implies (LR). Suppose x ∈ {a ∨ b, c, d}↓, that is, (H1) h ≤ x and (H2) a ∧ b ≤ x. In order to get our goal, that is, x ∈ {(h, a) ↓ ∩ (h, b) ↓)↑, let us suppose that (S) y ∈ {(h, a) ↓ ∩ (h, b) ↓)↑ and try to derive y ≤ x. Now, (S) means that for all z ∈ J, if

z ∈ {(h, a) ↓ ∩ (h, b) ↓)↑, then z ∈ y, that is,

(Y1) for all z ∈ J, if h ≤ z and a ≤ z, then y ≤ z and

(Y2) for all z ∈ J, if h ≤ z and b ≤ z, then y ≤ z.

Now, using (H), (Y1), (Y2), (H1) and (H2), we get our goal, that is, y ≤ x.

3.5 B-distributivity

The following definition seems to have appeared for the first time in [1, Theorem 2.2(i) p. 261]

Definition 5

A meet-semilattice J = (J; ≤) is B-distributive iff

(B) for all n, a₁, a₂, ..., aₙ, x ∈ J, if a₁ ∨ a₂ ∨ ... ∨ aₙ exists, then also (x ∧ a₁) ∨ (x ∧ a₂) ∨ ... ∨ (x ∧ aₙ) exists and equal x ∧ (a₁ ∨ a₂ ∨ ... ∨ aₙ).

We have the following fact.

Proposition 5

Let J = (J; ≤) be a meet—semilattice. Then, H-distributivity implies B-distributivity.

Proof

Let us have a H-distributivity meet semilattice J and let us take a, b, x ∈ J (the general case follows by induction). Let us suppose that a ∨ b exists in J. Then, also x ∧ (a ∨ b) exists in J. Our goal is to see that x ∧ (a ∨ b) = sup{x ∧ a, x ∧ b}. It is clear that x ∧ a, x ∧ b ≤ x ∧ (a ∨ b). Now, suppose both (F1) x ∧ b ≤ y and (F2) x ∧ a ≤ y. We have to see that x ∧ (a ∨ b) ≤ y. It immediately follows that

(X1) for all w ∈ J, if x ∧ b ≤ w and x ≤ w, then x ∧ (a ∨ b) ≤ w

Now, suppose that (F3) x ∧ b ≤ w and (F4) a ≤ w. Then, we have both
(X1') for all y ∈ J, if a ≤ y and x ≤ y, then x ∧ (a ∨ b) ≤ y, and

(X2') for all y ∈ J, if a ≤ y and b ≤ y, then x ∧ (a ∨ b) ≤ y

So, applying H-distributivity to (F3), (F4), (z1') and (X2'), we have x ∧ (a ∨ b) ≤ w, That is, we have proved.

(X2) for all w ∈ J, if x ∧ b ≤ w and a ≤ w, then x ∧ (a ∨ b) ≤ w.

Using H-distributivity, (F1), (F2), (X1) and (X2), it finally follows that x ∧ (a ∨ b) ≤ y as desired.

3.6 S_n-distributivity

The following definition seems to have appeared for the first time in [14]

Definition 6

A meet-semilattice J = (J; ≤) is said to be S_n – distributive for n a natural number, if

n ≤ 2, if f

(S_n) for all a_1, a_2, ... a_n, x ∈ J, if a_1 ∨ a_2 ∨ ... ∨ a_n exists, then also (x ∧ a_1) ∨ (x ∧ a_2) ... ∨ (x ∧ a_n) exists and equals x ∧ (a_1 ∨ a_2 ∨ ... ∨ a_n)

It is easy to see that B-distributivity implies S_n – distributivity, for any 2 ≤ n. It is also clear that for any 2 ≤ n, S_{n+1} implies S_n. On the other hand, we have that for no natural 2 ≤ n it holds that S_n – distributivity implies B-distributivity. In fact, it was proved that for any 2 ≤ n, does not imply S_{n+1} (see[12]), where infinite models using the real numbers are provided. As in the case of GS-distributivity and H-distributivity, it is natural to ask whether, for example finite models are possible. As in the cases just mentioned, the answer is negative as already proved in [16, theorem 7.1, p.1071]. In [15, theorem, p. 26] it is also proved that it is not possible to find infinite well-founded models.

Therefore, so far we have seen that, in the case of a meet-semilattice, we have the following chain of implications:

(GS) ⇒ (K) ⇒ (H) ⇒ (LR) ⇔ (ND) ⇒ (B) ... (S_n) ⇒ (S_{n-1}) ... (S_2).

Conclusion

In this paper we have proposed a notion of distributivity for meet-semilattices desired related to Gentzen’s conjunction elimination rule.

There are a number of open problems that we plan to address as a future research. In particular we can mention the following

- Distributive lattices are characterized by their lattice of ideals. In the case of meet-semilattices there are similar characterizations for GS-, K- and H-distributivity, but not for B- and S_n distributivity. The question in whether B- and S_n distributive meet-semilattices can be characterized by means of their ideals.

References