FLEXURAL ANALYSIS OF THICK BEAMS USING TRIGONOMETRIC SHEAR DEFORMATION THEORY

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Abstract *– In this present study, A trigonometric shear deformation theory is developed for flexural analysis of beams, in which number of variables are same as that in first-order shear deformation theory. The sinusoidal function is used in displacement field in terms of thickness coordinate to represent the shear deformation effect and satisfy the zero transverse shear stress condition at top and bottom surface of the beam. The Governing differential equation and boundary condition of the theory are obtained by using principle of virtual work. The fixed beam subjected to uniformly distributed load is examined using present theory. The numerical results obtained are compared with those of Elementary, Timoshenko and Higher-order shear deformation theory and the available solution in the literature.*

Key Words**: Trigonometric shear deformation theory, Transverse shear stresses, Flexure, Principle of virtual work, Axial shear stress, Equilibrium equation, Thick beam, Displacement.**

1. INTRODUCTION

It is well-known that elementary theory of bending of beam based on Euler-Bernoulli hypothesis that the plane sections which are perpendicular to the neutral axis before bending remain plane and perpendicular to the neutral axis after bending, implying that the transverse shear and transverse normal strains are zero. Thus, the theory disregards the effects of the shear deformation. It is also known as classical beam theory. The theory is applicable to slender beams and should not be applied to thick or deep beams. When elementary theory of beam (ETB) is used for the analysis thick beams, deflections are underestimated and natural frequencies and buckling loads are overestimated. This is the consequence of neglecting transverse shear deformations in ETB.

Bresse [1], Rayleigh [2] and Timoshenko [3] were the pioneer investigators to include refined effects such as rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires shear correction factor to appropriately represent the strain energy of deformation. To remove the discrepancies in classical and first order shear deformation theories, higher order or refined shear deformation theories were developed and are available in the open literature for static and vibration analysis of beam. Levinson [4], Bickford [5], Rehfield and Murty [6], Krishna Murty [7], presented parabolic shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor. There is another class of refined theories, which includes trigonometric function to represent the shear deformation effects through the thickness. Vlasov and Leontev [8], Stein [9] developed refined shear deformation theories for thick beams including sinusoidal function in terms of thickness coordinate in displacement field. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. Further Ghugal and Dahake [10] developed a trigonometric shear deformation theory for flexure of thick beam or deep beams taking into account transverse shear deformation effect. The number of variables in the present theory is same as that in the first order shear deformation theory. The trigonometric function is used in displacement field in terms of thickness coordinate to represent the shear deformation effects. A study of literature by Ghugal and Shimpi [11] indicates that the research work dealing with flexural analysis of thick beams using refined trigonometric and hyperbolic shear deformation theories are very scarce and is still in infancy. In this paper, a trigonometric shear deformation theory is developed for flexural analysis of thick beams. The theory is applied to a fixed beam to analysed the axial displacement, Transverse displacement, axial bending stress and transverse shear stress. The numerical results have been computed for various length to thickness ratios of the beams and the results obtained are compared with those of Elementary, Timoshenko, and higher-order shear deformation theory and with the available solution in the literature.

2. Formulation of problem

The beam under consideration as shown in Fig. (1). occupies in O-x-y-z Cartesian coordinate system the region:

$0 \le x \le L$ $\quad - b/2 \le y \le b/2$ $\quad - h/2 \le z \le h/2$

where x, y, z are Cartesian coordinates, L and b are the length and width in the x and y directions respectively, and h is the thickness of the beam. The beam is made up of homogeneous, linearly elastic isotropic material with the principal material axes parallel to the x and y axes in the plane of beam. The plate material obey's generalized Hook's law.

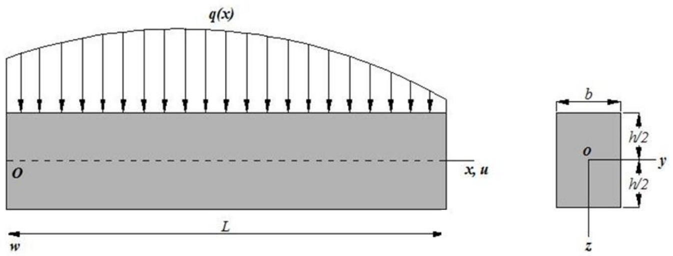

Figure1: Beam under bending in x-z plane

2.1 Assumptions made in the theoretical formulation

1. The axial displacement (u) consist of two parts:
   (a) Displacement given by elementary theory of bending.
   (b) Displacement due to shear deformation, which is assumed to be hyperbolic in nature with respect to thickness coordinate, such that maximum shear stress occurs at neutral axis as predicted by the elementary theory of bending of beam.
2. The axial displacement (u) is such that the resultant of in plane stress (σx) acting over the cross-section should result in only bending moment and should not in force in x direction.
3. The transverse displacement (w) in z direction is assumed to be function of x coordinate.
4. The displacements are small as compared to beam thickness.
5. The body forces are ignored in the analysis. (The body forces can be effectively taken into account by adding them to the external forces.)
6. One dimensional constitutive law' are used.
7. The beam is subjected to lateral load only.

2.2 The Displacement Field

Based on the above mention assumptions, the displacement field of the present beam theory can be expressed as follows.

$$u(x,z) = -z\frac{\partial w}{\partial x} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x) \tag{1}$$

$$w(x,z) = w(x) \tag{2}$$

Where,
u = Axial displacement in x direction which is function of x and z.
w = Transverse displacement in z direction which is function of x.
ϕ = Rotation of cross section of beam at neutral axis which is function of x.

Normal Strain:

$$\in_X = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \left[\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\right]\frac{\partial \phi}{\partial x} \tag{3}$$

Shear strain:

$$\gamma_X = \left[\cos\left(\frac{\pi z}{h}\right)\right]\phi \tag{4}$$

Stresses:

$$\sigma_X = E\varepsilon_X = -zE\frac{\partial^2 w}{\partial x^2} + \left[E\frac{h}{\pi}\sin\frac{\pi z}{h}\right]\frac{\partial \phi}{\partial x} \tag{5}$$

$$\tau_X = G\left[\cos\left(\frac{\pi z}{h}\right)\right]\phi \tag{6}$$

Where E and G after elastic constant of the beam material.

2.3 Governing Differential Equation

Governing differential equations and boundary conditions are obtained from Principle of virtual work. Using equations for stresses, strains and principle of virtual work, variationally consistent differential equations for beam under consideration are obtained. The principle of virtual work when applied to beam leads to:

$$b\int_{x=0}^{x=L}\int_{Z=-\frac{h}{2}}^{z=\frac{h}{2}}\left(\sigma_x\delta\varepsilon_x + \tau_{xz}\delta\gamma_{xz}\right)dxdz - \int_{x=0}^{x=L} q\delta w dx = 0 \tag{7}$$

Where δ = variational operator

Employing Green's theorem in equation (7) successively we obtain the coupled Euler Lagrange's equations which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows.

$$EI\frac{\partial^4 w}{\partial x^4}-A_0\frac{\partial^3\phi}{\partial x^3}=q(x) \tag{8}$$

$$EI\left(A_0\frac{\partial^3 w}{\partial x^3}-B_0\frac{\partial^2\phi}{\partial x^2}\right)+GAC_0\phi \tag{9}$$

Where A_0, B_0 and C_0 are the stiffness coefficients in governing equations. The associated consistent natural boundary conditions obtained are of following form along the edges x = 0 and x = L.

$$V_x=EI\left(\frac{\partial^3 w}{\partial x^3}-A_0\frac{\partial^2\phi}{\partial x^2}\right)=0 \tag{10}$$

Where w is prescribed

$$M_x=EI\left(\frac{\partial^2 w}{\partial x^2}-A_0\frac{\partial\phi}{\partial x}\right)=0 \tag{11}$$

Where $\frac{dw}{dx}$ is prescribed.

$$M_x=EI\left(A_0\frac{\partial^2 w}{\partial x^2}-B_0\frac{\partial\phi}{\partial x}\right)=0 \tag{12}$$

Where ϕ is Prescribed.

2.4 The General solution of Governing equilibrium equations of beam:

The general solution for transverse displacement w(x) and ϕ (x) can be obtained from Eqn. (8) and (9) by discarding the terms containing time (t) derivatives. Integrating and rearranging the Eqn. (8), we obtained the following equation,

$$\frac{\partial^3 w}{\partial x^3}=A_0\frac{\partial^2\phi}{\partial x^2}+\frac{Q(x)}{D} \tag{13}$$

where, Q(x) is generalized shear force for beam and it is given by,

$$Q(x)=\int_0^x q\,dx+C_1 \tag{14}$$

And by rearranging second governing Eqn. (9) the following equation is obtained.

$$\frac{\partial^3 w}{\partial x^3}=\frac{B_0}{A_0}\frac{\partial^2\phi}{\partial x^2}-\beta\phi \tag{15}$$

Now a single equation in terms of ϕ is obtained, by putting the Eqn. (3.8) in second governing Eqn. (15)

$$\alpha\frac{\partial^2\phi}{\partial x^2}-\beta(\phi)=\frac{Q(x)}{EI} \tag{16}$$

$$\phi=C_2\cosh(\lambda x)+C_3\sinh(\lambda x)-\left[\frac{Q(x)}{\beta EI}\right] \tag{17}$$

The equation of transverse displacement w(x) is obtained by substituting the expression of ϕ (x) in Eqn. (15) and integrating it thrice with respect to x. The general solution for w(x) is obtained as follows:

$$EIw(x)=\iiiint q\,dxdydz+\frac{C_1x^3}{6}+\left(\frac{\pi}{4}\lambda^2-\beta\right)\frac{EI}{\lambda^3}\left(C_2\sinh\lambda x+C_3\cosh\lambda x\right)+\frac{C_4x^2}{2}+C_5x+C_6 \tag{18}$$

where C1, C2, C3, C4, C5 and C6 are the constants of integration and can be obtained by imposing natural (forced) and kinematic boundary conditions of beams.

3. Illustrative Example

In order to prove the efficiency of the present theory, the following numerical examples are considered. The following material properties for beam are used. Material properties:

1. Modulus of Elasticity E = 210 GPa
2. Poisson's ratio μ = 0.30
3. Density = 7800 kg/m3

A. fixed beam with load $q(x)=q_0$

The beam has its origin on left hand side support and is fixed at *x* = 0 and *x* = *L*. The beam is subjected to distributed load, q_0 on surface z = +h/2 acting in downward z direction with minimum intensity of load q_0

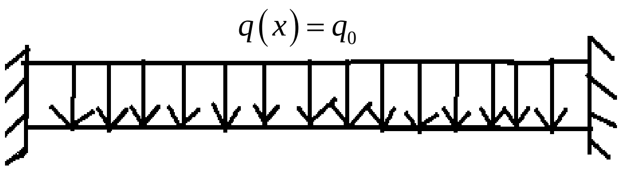

Figure 2: Fixed beam with load q_0

Boundary conditions associated with this problem are as follows:

At fixed end: x=0, L

$$\frac{\partial w}{\partial x}=\phi=w=0$$

General Expressions obtained for w(x) and ϕ(x) are as follows.

$$w(x)=\left[\begin{array}{l}5\dfrac{E}{G}\dfrac{h^2}{L^2}\dfrac{A_0^2}{C_0}\left(\dfrac{\cosh\lambda x-\sinh\lambda x-1}{\lambda L}+\dfrac{x}{L}-\dfrac{1}{2}\dfrac{x^2}{L^2}\right)\\ +\left(\dfrac{5x^4}{L^4}+\dfrac{5x^2}{L^2}-\dfrac{10x^3}{L^3}\right)\end{array}\right] \tag{19}$$

$$\phi(x)=\left[\frac{A_0}{C_0}\frac{q_0}{Gb}\frac{L}{h}\frac{1}{2}\left(\sinh\lambda x-\cosh\lambda x+1-\frac{2x}{L}\right)\right] \tag{20}$$

Expression for Non-Dimensional Axial Displacement $(\overline{u})$, Axial Stress $(\overline{\sigma_x})$, Maximum Transverse Shear Stresses $(\overline{\tau_{zx}^{CR}})$ and $(\overline{\tau_{zx}^{EE}})$ are as follows.

$$\bar{u}=-\frac{z}{h}\frac{L^3}{h^3}\left\{\begin{array}{l}\left\{\begin{array}{l}\left[2\dfrac{x^3}{L^3}-3\dfrac{x^2}{L^2}+\dfrac{x}{L}\right]+\\ \left[\dfrac{1}{2}\dfrac{E}{G}\dfrac{A_0^2}{C_0}\dfrac{h^2}{L^2}\left(\sinh\lambda x-\cosh\lambda x+1-\dfrac{x}{L}\right)\right]+\end{array}\right\}\\ +\dfrac{A_0}{C_0}\dfrac{1}{2}\dfrac{E}{G}\dfrac{L}{h}\left(\dfrac{h}{\pi}\sin\dfrac{\pi z}{h}\right)\left(\sinh\lambda x-\cosh\lambda x+1-2\dfrac{x}{L}\right)\end{array}\right\} \tag{21}$$

$$\overline{\sigma_x}=-\frac{z}{h}\frac{L^2}{h^2}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(6\dfrac{x^2}{L^2}-6\dfrac{x}{L}+1\right)\\ +\dfrac{1}{2}\dfrac{A_0^2}{C_0}\dfrac{E}{G}\dfrac{h^2}{L^2}\left(\lambda L\cosh\lambda x-\lambda L\sinh\lambda x-1\right)\end{array}\right\}\\ +\dfrac{A_0}{C_0}\dfrac{E}{G}\dfrac{1}{2}\left[\dfrac{h}{\pi}\sin\dfrac{\pi z}{h}\right]\left(\lambda L\cosh\lambda x-\lambda L\sinh\lambda x-2\right)\end{array}\right\} \tag{22}$$

$$\overline{\tau_{zx}^{CR}}=\left[\frac{A_0}{C_0}\frac{L}{h}\frac{1}{2}\left(\cos\frac{\pi z}{h}\right)\left[\sinh\lambda x-\cosh\lambda x+1-2\frac{x}{L}\right]\right] \tag{23}$$

$$\bar{\tau}_{zx}^{EE}=\frac{1}{8}\frac{L}{h}\left(\frac{4z^2}{h^2}-1\right)\left\{\begin{array}{l}\left\{\left[12\dfrac{x}{L}-6\right]+\left[\dfrac{1}{2}\dfrac{A_0^2}{C_0}\dfrac{E}{G}\dfrac{h^2}{L^2}\left(\lambda^2L^2\sinh\lambda x-\lambda^2L^2\cosh\lambda x\right)\right]\right\}\\ +\dfrac{A_0}{C_0}\dfrac{E}{G}\dfrac{1}{2}\dfrac{h}{L}\left[\dfrac{h}{\pi}\cos\dfrac{\pi}{2}\right]\left(\lambda^2L^2\sinh\lambda x-\lambda^2L^2\cosh\lambda x\right)\end{array}\right\} \tag{24}$$

4. Numerical Result

The numerical results for axial displacements, transverse displacements, bending stress and transverse shear stress are presented in following non dimensional form and the values are presented in table 1 and table 2.

$$\bar{w}=\frac{10Ebh^3}{q_0L^4}w \qquad \bar{u}=\frac{Eb}{q_0h}u$$

$$\overline{\sigma_x}=\frac{b}{q_0}\sigma_x \qquad \overline{\tau_{zx}}=\frac{b}{q_0}\tau_{zx}$$

Table 1: Non-Dimensional Axial Displacement $(\overline{u})$ at (x =0.75L, z = h/2), Transverse Deflection $(\overline{w})$ at (x = 0.75L, z = 0.0), Axial Stress $(\overline{\sigma_x})$ at (x = 0, z = h/2), Maximum Transverse Shear Stresses $(\overline{\tau_{zx}^{CR}})$ and $(\overline{\tau_{zx}^{EE}})$ at (x = 0:01L, z = h/2) of Fixed Beam Subjected to Varying Load for Aspect Ratio 4.

Source	Model	$\bar{w}$	$\bar{u}$	$\overline{\sigma_x}$	$\overline{\tau_{zx}^{CR}}$	$\overline{\tau_{zx}^{EE}}$
Present	TSDT	3.48	2.65	-2.68	1.58	97.96
Dahake	HSDT	3.48	4.21	-2.52	0.08	142.07
Timoshenko	FSDT	2.00	1.04	-2.78	0.23	2.94
Bernoulli-Euler	ETB	0.17	1.04	-2.78	-	2.94

Table 2: Non-Dimensional Axial Displacement $(\overline{u})$ at (x =0.75L, z = h/2), Transverse Deflection $(\overline{w})$ at (x = 0.75L, z = 0.0), Axial Stress $(\overline{\sigma_x})$ at (x = 0, z = h/2), Maximum Transverse Shear Stresses $(\overline{\tau_{zx}^{CR}})$ and $(\overline{\tau_{zx}^{EE}})$ at (x = 0:01L, z = h/2) of Fixed Beam Subjected to Varying Load for Aspect Ratio 10.

Source	Model	$\bar{w}$	$\bar{u}$	$\overline{\sigma_x}$	$\overline{\tau_{zx}^{CR}}$	$\overline{\tau_{zx}^{EE}}$
Present	TSDT	3.48	1.18	-1.71	3.10	54.51
Dahake	HSDT	3.48	1.18	-1.71	0.02	72.85
Timoshenko	FSDT	2.00	1.63	-1.73	3.73	7.35
Bernoulli-Euler	ETB	0.17	1.63	-1.73	-	7.35

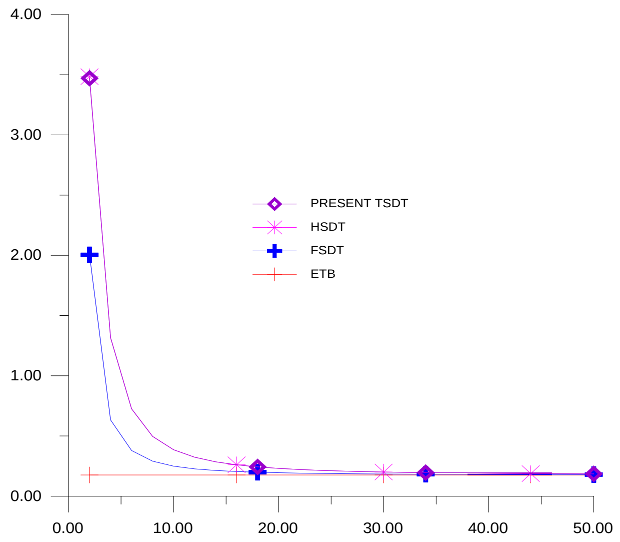

Fig 3: Variation of transverse displacements w

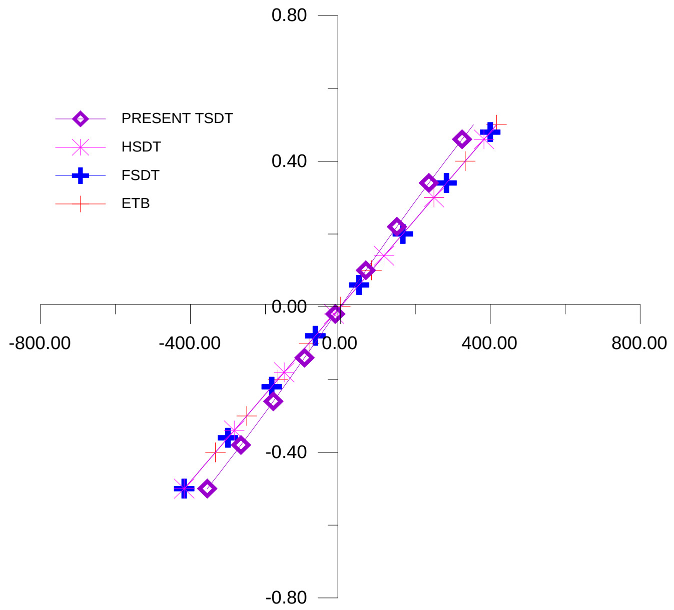

Fig 5: Variation of Maximum Axial displacement u for AS 10

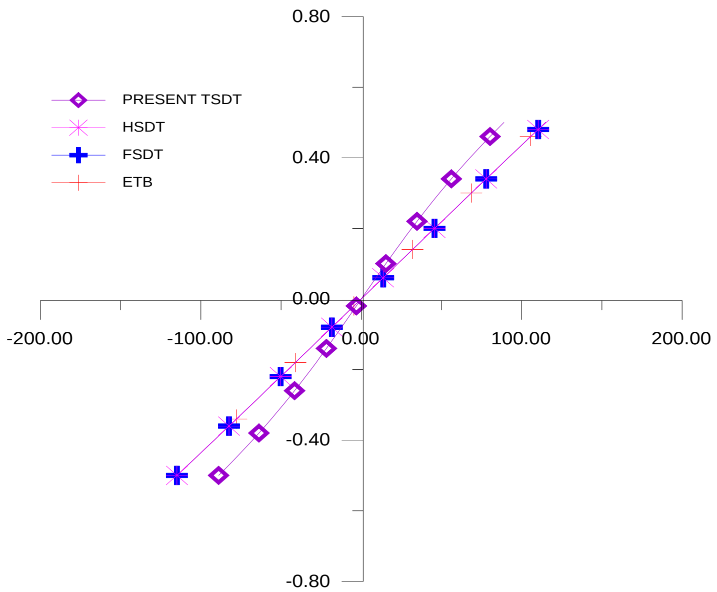

Fig 4: Variation of Maximum Axial displacement u for AS 04

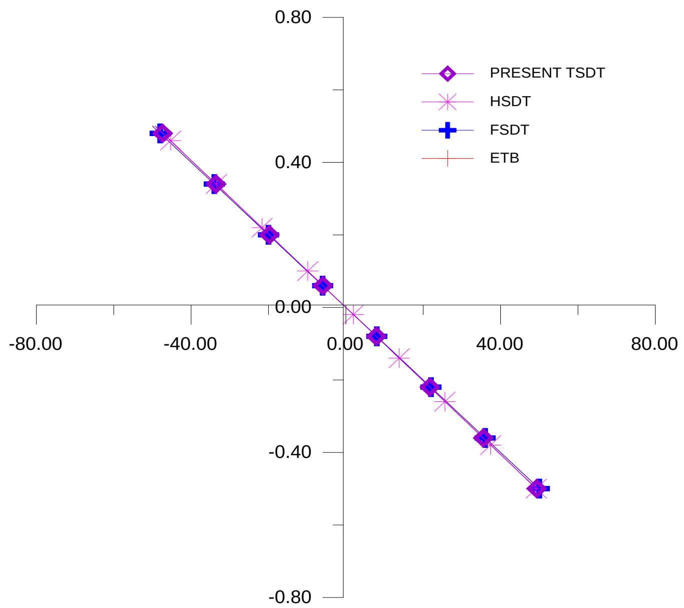

Fig 6: Variation of maximum axial stress $\overline{\sigma_x}$ For AS 04

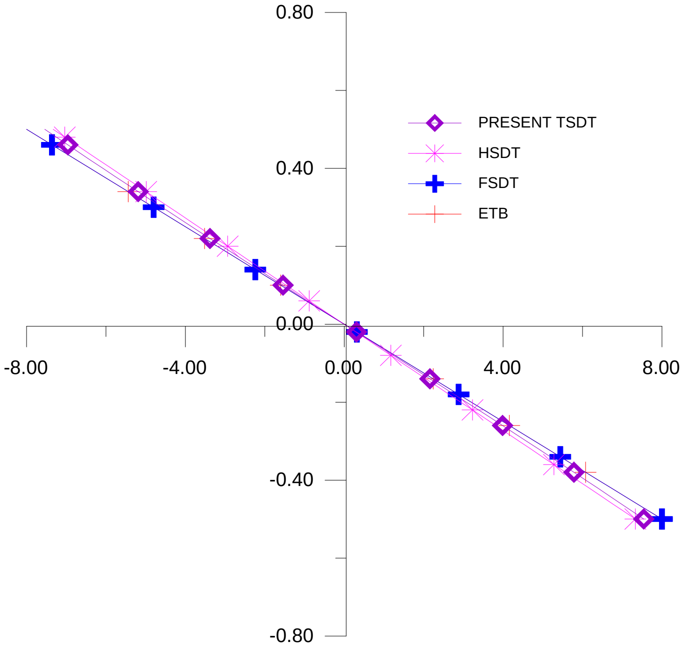

Fig 7: Variation of maximum axial stress $\overline{\sigma_x}$ For AS 10

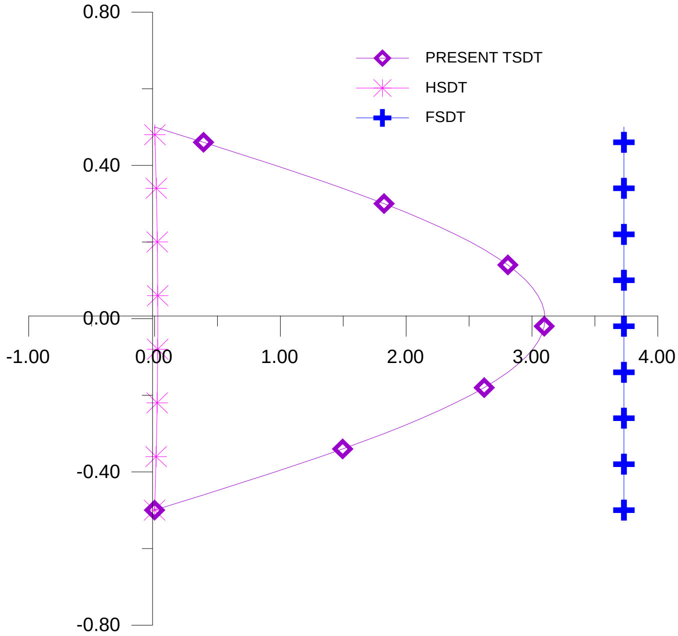

Fig 9: Variation of transverse shear stress $\overline{\tau_{zx}^{CR}}$ for AS 10

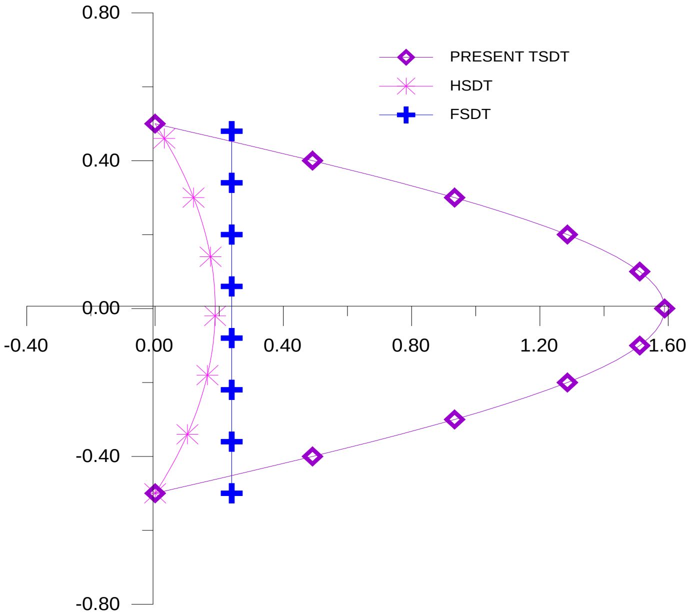

Fig 8: Variation of trans verse shear stress $\overline{\tau_{zx}^{CR}}$ for AS 04

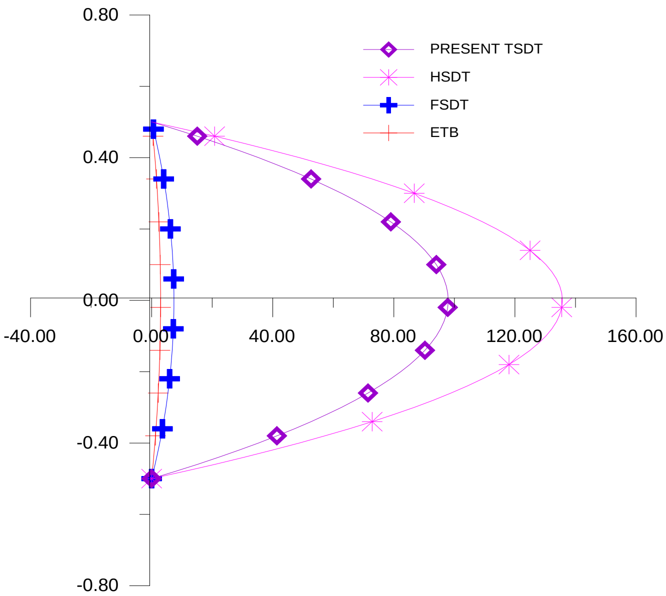

Fig 10: Variation of transverse shear stress $\overline{\tau_{zx}^{EE}}$ for AS 04

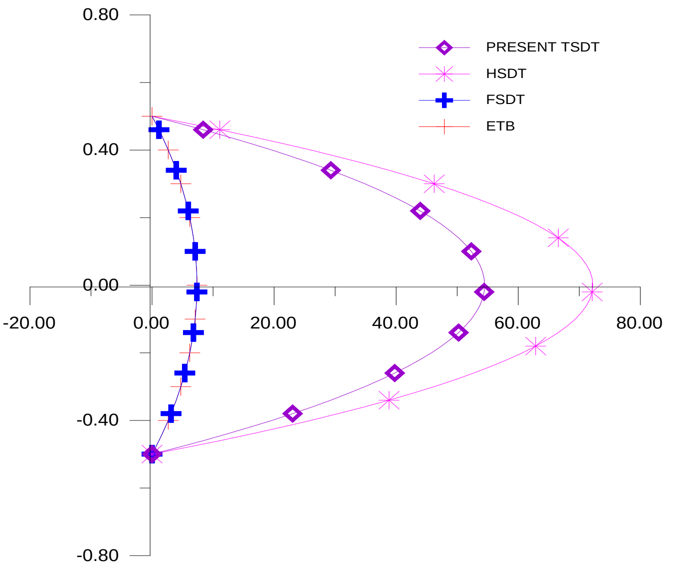

Fig 11: Variation of transverse shear stress $\overline{\tau_{zx}^{EE}}$ for AS 10

5. CONCLUSIONS

From the static flexural analysis of fixed beam following conclusions are drawn:

1. The result of maximum transverse displacement $\overline{w}$ obtained by present theory is in excellent agreement with those of other equivalent refined theories. The variation of AS 04 and AS 10 are present as shown in fig. 3
2. From figure 4 and 5, it can be observed that, the result of axial displacement $\overline{u}$ for beam subjected to uniformly load varies linearly through the thickness of beam of AS 04 and AS 10 respectively.
3. The maximum Non-dimensional axial stress $\overline{\sigma_x}$ For AS 06 and AS 7 varies linearly through the thickness of beam as shown in fig.10 and fig. 11 respectively.
4. The transverse shear stress $\overline{\tau_{zx}^{CR}}$ and $\overline{\tau_{zx}^{EE}}$ are obtained directly by constitutive relation. Fig. 8, 9, 10, and 11. Shows the through thickness variation of transverse shear stress for thick beam for AS 04 and AS 10. From this fig. it can be observed that, the transverse shear stress satisfies the zero condition at top and at bottom surface of the beam.

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