Characteristic Exponent of Collinear Libration Point in \( L_2 \) in Photo-gravitational Restricted Problem of 2+2 Bodies When Bigger Primary is a Triaxial Rigid Body Perturbed by Coriolis and Centrifugal Forces

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Abstract - Stability of collinear libration point \( L_2 \) in photo-gravitational restricted problem of 2+2 bodies when bigger primary is a triaxial rigid body perturbed by coriolis and centrifugal forces has been studied in which it is found that \( L_2 \) is unstable.

Key Words: Stability, Collinear libration point, Photo-gravitation, Triaxial rigid body, Coriolis force, Centrifugal force.

1. INTRODUCTION

Equilibrium solutions of restricted problem of 2+2 bodies are derived by Whipple (1984) in which \( M_1 \) and \( M_2 \) are two point masses moving in the circular Keplerian orbit about their centre of mass. He assumed that \( M_1 \geq M_2 \). Two minor bodies \( m_1 \) and \( m_2 \) (\( m_1, m_2 \ll M_2 \)) move in the gravitational fields of primaries \( (M_1 \text{ and } M_2) \). They attract each other but do not perturb the primaries. He showed the existence of fourteen equilibrium solutions. Six of these solutions are located about the collinear Lagrangian points of classical restricted problem of three bodies, eight solutions are found in the neighborhood of triangular Lagrangian points.

Sharma, Taqvi and Bhatnagar (2001) studied the existence and stability of libration points in the restricted three body problem when primaries are triaxial rigid body and source of radiation. They found five libration points, two triangular and three collinear they also observed that collinear points are unstable while triangular libration points are stable for the mass parameter \( 0 \leq \mu < \mu_{\text{crit}} \).

Garain and Chakraborty (2007) derived libration points and examined stability in Robe’s three body restricted problem when second primary is a triaxial rigid body perturbed by coriolis and centrifugal forces. They found that the collinear libration points are deviated due to perturbation of centrifugal forces and triaxility of the second primary. Perturbation of coriolis force and triaxility character of the body play important role for finding the region of stability.

Hoque and Garain (2014) computed collinear libration point \( L_2 \). In the case of 2+2 body problem when perturbation effects act in coriolis and centrifugal forces, small primary is a radiating body and bigger primary as a triaxial rigid body.

2. EQUATIONS OF MOTION

Whipple’s (1984) equation of motion of restricted problem of 2+2 bodies in synodic system be

\[
\ddot{x}_i - 2\dot{y}_i = \frac{\partial T}{\partial x_i} \tag{1}
\]

\[
\ddot{y}_i + 2\dot{x}_i = \frac{\partial T}{\partial y_i} \tag{2}
\]

\[
\ddot{z}_i = \frac{\partial T}{\partial z_i}, \quad (i = 1, 2) \tag{3}
\]
\[
T = \sum_{i=1}^{3} \mu_i \left[ \frac{r_i^2}{2} + \frac{1}{r_i} + \frac{r_i^2}{2r} \right]
\]
\[
\mu = \frac{M_2}{M_1 + M_2}, \quad \mu_i = \frac{m_i}{M_1 + M_2}, \quad (i = 1, 2), \quad r_i^2 = (x_i - \mu)^2 + y_i^2 + z_i^2,
\]
\[
r_i^2 = (x_i - \mu + 1)^2 + y_i^2 + z_i^2
\]

Here we consider \( M_2 \), a radiating body and \( M_1 \), a triaxial rigid body, we have also considered effect of perturbation in coriolis and centrifugal forces in this configuration.

Hence force function \( T \) reduces to \( U \) as follows:
\[
U = \sum_{i=1}^{3} \left[ \frac{1}{2} \beta (x_i^2 + y_i^2) + \frac{1}{r_i} + \frac{r_i^2}{2r} + (1-\mu) + \frac{\mu}{r_i} + \frac{r_i^2}{2r} \right] - \frac{3(1-\mu)(\sigma_1 - \sigma_2)y_i^2}{2r_i^3}
\]
Where, \( q = 1-\epsilon, \beta = 1+\epsilon' \) and \( i = 1, 2 \).

Equilibrium points of the system are those points where \( \dot{x}_i = \dot{y}_i = \dot{z}_i = \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial y_i} = \frac{\partial U}{\partial z_i} = 0, \quad (i = 1, 2) \).

Thus we have,
\[
\beta x_i - \frac{(1-\mu)(x_i - \mu)}{r_i^3} - \mu(x_i - x_2) - \frac{3(1-\mu)(\sigma_1 - \sigma_2)(x_i - \mu)}{2r_1} + \frac{15(1-\mu)(\sigma_1 - \sigma_2)(x_i - \mu)y_i^2}{2r_1^3} = 0
\]
\[
\beta y_i - \frac{(1-\mu)y_i}{r_i^3} - \mu y_i - \frac{3(1-\mu)(\sigma_1 - \sigma_2)y_i}{2r_1} - \frac{3(1-\mu)(\sigma_1 - \sigma_2)(y_i - \mu)}{2r_1} = 0
\]
\[
\beta z_i - \frac{(1-\mu)z_i}{r_i^3} - \mu z_i - \frac{3(1-\mu)(\sigma_1 - \sigma_2)z_i}{2r_1} + \frac{15(1-\mu)(\sigma_1 - \sigma_2)z_iy_i^2}{2r_1^3} = 0
\]

From equations (7) and (10), we have \( z_1 = z_2 = 0 \).

By inspection, it can be seen that equations (6) and (9) are satisfied when \( y_1 = y_2 = 0 \).

Now we have to determine \( x_1 \) and \( x_2 \) such that the following simplified forms of equations (5) and (8) are satisfied.
\[ \beta x_1 \left( \frac{1 - \mu}{x_1 - \mu} \right) - \frac{q \mu (x_1 - \mu + 1)}{|x_1 - \mu|^3} - \frac{\mu_2 (x_1 - x_2)}{|x_1 - x_2|^3} \]
\[ - \frac{3(1 - \mu)2(\sigma_1 - \sigma_2)(x_1 - \mu)}{2|x_1 - \mu|^5} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2)(x_1 - \mu)y_1^2}{2|x_1 - \mu|^7} = 0 \]  

(11)

And

\[ \beta x_2 \left( \frac{1 - \mu}{x_2 - \mu} \right) - \frac{q \mu (x_2 - \mu + 1)}{|x_2 - \mu|^3} - \frac{\mu_2 (x_2 - x_1)}{|x_2 - x_1|^3} \]
\[ - \frac{3(1 - \mu)2(\sigma_1 - \sigma_2)(x_2 - \mu)}{2|x_2 - \mu|^5} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2)(x_2 - \mu)y_2^2}{2|x_2 - \mu|^7} = 0 \]  

(12)

The solution of equations (11) and (12) can be obtained with the help of power series.

Let \( \epsilon_i = \frac{\mu_i}{(\mu_1 + \mu_2)^3} \), \( i = 1, 2 \)  

(13)

\[ : \epsilon_1 = \frac{\mu_1}{(\mu_1 + \mu_2)^3} \quad \text{and} \quad \epsilon_2 = \frac{\mu_2}{(\mu_1 + \mu_2)^3} \]  

(14)

\[ \therefore \mu_1 \epsilon_1 = \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^3} = \mu_1 \epsilon_2 = k \text{(say)} \]  

(15)

Let \( x_i = L_i' + \sum_{j=1}^{n} a_{ij} \epsilon_j^i \) for \( i = 1, 2, 3 \)  

(16)

Where \( L_1', L_2', L_3' \) are equilibrium points in the photo-gravitational restricted problem of three bodies when a bigger primary is a triaxial rigid body perturbed by Coriolis and centrifugal forces and \( x_1 \) be the \( x \) coordinate of the first small body.

\[ x_2 = L_2' + \sum_{j=1}^{n} a_{2j} \epsilon_j^i \text{ for } i = 1, 2, 3 \]  

(17)

Similar to Whipple, \( x_i = L_i' + \frac{(\pm 1)}{2^{1/2}} \frac{\mu_2}{(\Omega_{\ast})^{1/2} (\mu_1 + \mu_2)^{3/2}} \) \( \text{where } i = 1, 2, 3 \)  

(18)

And

\[ x_2 = L_2' - \frac{(\pm 1)}{2^{1/2}} \frac{\mu_1}{(\Omega_{\ast})^{1/2} (\mu_1 + \mu_2)^{3/2}} \text{ where } i = 1, 2, 3 \]  

(19)

Hoque and Garain (2014) obtained two values \((x_1, 0, 0)\) and \((x_2, 0, 0)\) of \( L_2 \) in which

\[ x_1 = a_{21} - b_{21} e + c_{21} e' - 2d_{21} \sigma_1 + d_{21} \sigma_2 \]

and

\[ x_2 = a_{22} - b_{22} e + c_{22} e' - 2d_{22} \sigma_1 + d_{22} \sigma_2 \]
Where

\[ a_{21} = \mu - 1 + a_2 + \frac{(\pm 1)\mu_2}{A_2^{\frac{1}{2}}(\mu_1 + \mu_2)^{\frac{1}{2}}} \quad \text{and} \quad b_{21} = b_2 - \frac{(\pm 1)\mu_2B_2}{3A_2^{\frac{3}{2}}(\mu_1 + \mu_2)^{\frac{3}{2}}} \]

\[ a_{22} = \mu - 1 + a_2 - \frac{(\pm 1)\mu_1}{A_2^{\frac{1}{2}}(\mu_1 + \mu_2)^{\frac{1}{2}}} , \quad b_{22} = b_2 + \frac{(\pm 1)\mu_1B_2}{3A_2^{\frac{3}{2}}(\mu_1 + \mu_2)^{\frac{3}{2}}} \]

\[ c_{22} = c_2 + \frac{(\pm 1)\mu_1C_2}{3A_2^{\frac{3}{2}}(\mu_1 + \mu_2)^{\frac{3}{2}}} \quad \text{and} \quad d_{22} = d_2 - \frac{(\pm 1)\mu_1D_2}{3A_2^{\frac{3}{2}}(\mu_1 + \mu_2)^{\frac{3}{2}}} \]

They obtained the particular values of \( x_1 \) and \( x_2 \) for different values of \( \mu \), \( \mu_1 \) and \( \mu_2 \).

Our characteristic equation corresponding to the point \((x_1, 0, 0)\) is

\[ f(\lambda) = \lambda^4 + \lambda^2 \left( 4 - \frac{U_{x,y} - U_{x,y}^e}{\mu_1} \right) + \frac{1}{\mu_1^2}(U_{x,y} - U_{x,y}^e) = 0 \]  \hspace{1cm} (20)

\[ \frac{U_{x,y}}{\mu_1} = \left[ \beta + \frac{2(1 - \mu)}{r_{11}^2} + \frac{2q\mu}{r_{21}^2} + \frac{2\mu_2}{r^3} + \frac{6(1 - \mu)(2\sigma_1 - \sigma_2)}{r_{11}^3} - \frac{45(1 - \mu)(\sigma_1 - \sigma_2)y_1^2}{r_{11}^3} \right] \]

\[ \frac{U_{x,y}}{\mu_1} = \left[ \beta - \frac{(1 - \mu)}{r_{11}^2} - \frac{q\mu}{r_{21}^2} - \frac{\mu_2}{r^3} - \frac{3(1 - \mu)(4\sigma_1 - 3\sigma_2)}{2r_{11}^3} + \frac{3(1 - \mu)y_1^2}{r_{21}^3} + \frac{3q\mu y_1^2}{r_{21}^3} \right] \]

\[ + \frac{3\mu_1(y_1 - y_2)^2}{r^5} + \frac{15(1 - \mu)(7\sigma_1 - 6\sigma_2)y_1^2}{2r_{11}^7} - \frac{105(1 - \mu)(\sigma_1 - \sigma_2)y_1^4}{2r_{11}^7} \]

\[ \frac{U_{x,y}}{\mu_1} = \frac{3(1 - \mu)(x_1 - \mu)y_1}{r_{11}^5} + \frac{3q\mu(x_1 - \mu + 1)y_1}{r_{21}^5} + \frac{3\mu_1(x_1 - x_2)(y_1 - y_2)}{r^5} \]

\[ + \frac{15(1 - \mu)(2\sigma_1 - \sigma_2)(x_1 - \mu)y_1}{2r_{11}^7} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2)(x_1 - \mu)y_1}{2r_{11}^7} \]

In this case, \( y_1 = 0 \) and \( y_2 = 0 \).

Therefore the above equations reduce to

\[ \frac{U_{x,y}}{\mu_1} = \left[ \beta + \frac{2(1 - \mu)}{(x_1 - \mu)^3} + \frac{2q\mu}{(x_1 - \mu + 1)^3} + \frac{2\mu_2}{(x_1 - x_2)^3} + \frac{6(1 - \mu)(2\sigma_1 - \sigma_2)}{(x_1 - \mu)^5} \right] \]

\[ \frac{U_{x,y}}{\mu_1} = 0 \]

And

\[ \frac{U_{x,y}}{\mu_1} = \left[ \beta - \frac{(1 - \mu)}{(x_1 - \mu)^3} - \frac{q\mu}{(x_1 - \mu + 1)^3} - \frac{\mu_2}{(x_1 - x_2)^3} - \frac{3(1 - \mu)(4\sigma_1 - 3\sigma_2)}{2(x_1 - \mu)^5} \right] \]

For the collinear equilibrium solutions the partial derivatives contained in equation (20) reduce to

\[ \frac{U_{x,y}}{\mu_1} = \left[ \beta + \frac{2(1 - \mu)}{|x_1 - \mu|} + \frac{2q\mu}{|x_1 - \mu + 1|} + \frac{2\mu_2}{|x_1 - x_2|} + \frac{6(1 - \mu)(2\sigma_1 - \sigma_2)}{|x_1 - \mu|^5} \right] \]
\[ \frac{U_{y_1 y_1}}{\mu_1} = 0 \]
\[ \frac{U_{y_1 y_1}}{\mu_1} = \left[ (1 + e') - \frac{(1 - \mu)}{|x_1 - \mu|^3} - \frac{(1 - \epsilon)\mu}{|x_1 - \mu + 1|^3} - \frac{\mu_2}{|x_1 - x_2|^3} - \frac{3(1 - \mu)(4\sigma_1 - 3\sigma_2)}{2|x_1 - \mu|^5} \right] \]

Let
\[ \frac{U_{y_1 y_1}}{\mu_1} = A_{21} - B_{21} \epsilon + C_{21} \epsilon' - D_{21} \sigma_1 + E_{21} \sigma_2 \]
\[ x_1 = a_{21} - b_{21} \epsilon + c_{21} \epsilon' - 2d_{21} \sigma_1 + d_{21} \sigma_2 \]
and
\[ x_2 = a_{22} - b_{22} \epsilon + c_{22} \epsilon' - 2d_{22} \sigma_1 + d_{22} \sigma_2 \]
\[ \Rightarrow \frac{U_{y_1 y_1}}{\mu_1} = A_{21} - B_{21} \epsilon + C_{21} \epsilon' - D_{21} \sigma_1 + E_{21} \sigma_2 \]

Where,
\[ A_{21} = 1 - \frac{(1 - \mu)}{|a_{21} - \mu|^3} - \frac{\mu}{|a_{21} - \mu + 1|^3} - \frac{\mu_2}{|a_{21} - a_{22}|^3}, \]
\[ B_{21} = \frac{3(1 - \mu)b_{21}}{|a_{21} - \mu|^4} + \frac{3\mu b_{21}}{|a_{21} - \mu + 1|^4} + \frac{3\mu_2(b_{21} - b_{22})}{|a_{21} - a_{22}|^4}, \]
\[ C_{21} = 1 + \frac{3(1 - \mu)c_{21}}{|a_{21} - \mu|^4} + \frac{3\mu c_{21}}{|a_{21} - \mu + 1|^4} + \frac{3\mu_2(c_{21} - c_{22})}{|a_{21} - a_{22}|^4}, \]
\[ D_{21} = \frac{6(1 - \mu)d_{21}}{|a_{21} - \mu|^4} + \frac{6\mu d_{21}}{|a_{21} - \mu + 1|^4} + \frac{6\mu_2(d_{21} - d_{22})}{|a_{21} - a_{22}|^4} + \frac{6(1 - \mu)}{|a_{21} - \mu|^5}, \]
and
\[ E_{21} = \frac{3(1 - \mu)d_{21}}{|a_{21} - \mu|^4} + \frac{3\mu d_{21}}{|a_{21} - \mu + 1|^4} + \frac{3\mu_2(d_{21} - d_{22})}{|a_{21} - a_{22}|^4} + \frac{9(1 - \mu)}{2|a_{21} - \mu|^5} \]

Here we see that \( U_{x_1 y_1} > 0 \) and \( U_{x_1 y_1} = 0 \). Therefore equation (20) reduces to

\[ f(\lambda) = \lambda^4 + \lambda^2 \left( 4 - \frac{1}{\mu_1} U_{x_1 y_1} - U_{y_1 y_1} \right) + \frac{1}{\mu_1^2} U_{x_1 y_1} U_{y_1 y_1} = 0 \]  \( \text{(20a)} \)

**Case I:** Let \( \lambda_1^2 \) and \( \lambda_2^2 \) be the roots of equation \( \text{(20a)} \).

**Sub case (i):** If \( \lambda_1^2 \) and \( \lambda_2^2 \) both are real and one of them is positive, let \( \lambda_1^2 \) is a positive quantity, then square root of \( \lambda_1^2 \) must be real and of opposite sign. In this case characteristic roots will be a real number. So in this case \( L_2 \) is unstable. We may get similar result in the case of \( \lambda_2^2 \).

**Sub case (ii):** If \( \lambda_1^2 \) and \( \lambda_2^2 \) both are real and negative, let \( \lambda_1^2 \) is a negative quantity, then two roots of \( \lambda_1^2 \) be purely imaginary. Similarly two roots of \( \lambda_2^2 \) be purely imaginary. In this case \( L_2 \) should be stable. Again when \( \lambda_1^2 \) and \( \lambda_2^2 \) both are negative quantity then \( \lambda_1^2 \lambda_2^2 \) is a positive quantity

\( \text{(20b)} \)

From equation \( \text{(20a)} \), we have obtained that
\[ \lambda_1^2 \lambda_2^2 = \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} = \text{a negative quantity } (\because U_{x_{i1}} > 0, U_{y_{j1}} < 0) \]  

(20c)

Since (20b) and (20c) contradict each other. So \( L_2 \) is unstable.

Now let \( 4 - \frac{U_{x_{i1}}}{\mu_1} - U_{y_{j1}} = A_{23} \)

**Case II:** if \( A_{23} > 0 \) i.e., \( 4 > \frac{1}{\mu_1} U_{x_{i1}} + U_{y_{j1}} \) then \( f(\lambda) = \lambda^4 + A_{23} \lambda^2 + \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} = 0 \).

**Sub case (i):** It is clear that \( U_{x_{i1}} > 0 \) and let \( U_{y_{j1}} > 0 \) then \( \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} > 0 \) so, \( f(\lambda) = 0 \) has no change of signs and as such it has no positive real roots. \( f(-\lambda) = \lambda^4 + A_{23} \lambda^2 + \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} = 0 \) also has no change of signs and as such it has no positive real roots i.e., \( f(\lambda) \) has no negative real roots. So in this case we can say that all roots of \( f(\lambda) = 0 \) are imaginary.

**Sub case (ii):** if \( U_{y_{j1}} < 0 \) then similar to Yadav’s case \( \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} < 0 \).

Let \( \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} = -B_{23}, B_{23} > 0 \), so, \( f(\lambda) = \lambda^4 + A_{23} \lambda^2 - B_{23} = 0 \). Here \( f(\lambda) = 0 \) has only one change of sign and as such it has one positive real root. \( f(-\lambda) = \lambda^4 + A_{23} \lambda^2 - B_{23} = 0 \) has only one change of sign and as such it has one positive real root i.e., \( f(\lambda) \) has one negative real root.

**Case III:** if \( A_{23} < 0 \) i.e., \( 4 < \frac{1}{\mu_1} U_{x_{i1}} + U_{y_{j1}} \) then let \( A_{23} = -C_{23}, C_{23} > 0 \) then

\[ f(\lambda) = \lambda^4 - C_{23} \lambda^2 + \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} = 0 \]

**Sub case (i):** Let \( U_{y_{j1}} > 0 \), then \( \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} > 0 \).

Then, \( f(\lambda) = 0 \) has two changes of signs and as such it has at most two positive real roots. \( f(-\lambda) = \lambda^4 - C_{23} \lambda^2 + \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} = 0 \) also has two changes of signs and as such it has at most two positive real roots i.e., \( f(\lambda) \) has at most two negative real roots.

**Sub case (ii):** if \( U_{y_{j1}} < 0 \), then \( \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} < 0 \).

Let \( \frac{1}{\mu_1} U_{x_{i1}} U_{y_{j1}} = -D_{23}, D_{23} > 0 \). Here \( f(\lambda) = \lambda^4 - C_{23} \lambda^2 - D_{23} = 0 \) has only one change of sign and as such it has one positive real root. \( f(-\lambda) = \lambda^4 - C_{23} \lambda^2 - D_{23} = 0 \) has only one change of sign and as such it has one positive real root i.e., \( f(\lambda) \) has one negative real root.

In both the cases we see that \( f(\lambda) = 0 \) has one positive and one negative real root. So the libration point \( L_2 \) is unstable.

The characteristic equation corresponding to the point \( (x_L, 0, 0) \) be

\[ f(\lambda) = \lambda^4 + \lambda^2 \left( 4 - \frac{1}{\mu_2} U_{x_2x_2} - U_{y_2y_2} \right) + \frac{1}{\mu_2^2} U_{x_2x_2} U_{y_2y_2} - U_{x_2y_2}^2 = 0 \]  

(21)

\[ U_{x_2x_2} = \frac{\beta + \frac{2(1-\mu)}{(x_2-\mu)^2} + \frac{2q\mu}{(x_2-\mu+1)^2} + \frac{2\mu_1}{(x_1-x_2)^2} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(x_2-\mu)^3}}{\mu_2} \]

\[ U_{y_2y_2} = \frac{\beta - \frac{1(1-\mu)}{(x_2-\mu)^3} - \frac{q\mu}{(x_2-\mu+1)^3} + \frac{\mu_1}{(x_1-x_2)^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2(x_2-\mu)^3}}{\mu_2} \]

\[ U_{x_2y_2} = 0 \]

\[ U_{x_2y_2} = \left[ \beta + \frac{2(1-\mu)}{|x_2-\mu|^2} + \frac{2q\mu}{|x_2-\mu+1|^2} + \frac{2\mu_1}{|x_1-x_2|^2} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{|x_2-\mu|^3} \right] \]

Let \[ \frac{U_{y_2y_2}}{\mu_2} = A_{22} - B_{22} \epsilon + C_{22} \epsilon' - D_{22} \sigma_1 + E_{22} \sigma_2 \]  

\[ x_1 = a_{22} - b_{22} \epsilon + c_{22} \epsilon' - 2d_{22} \sigma_1 + d_{22} \sigma_2 \]

Here we see that \[ U_{x_2x_2} > 0 \] and \[ U_{x_2y_2} = 0 \]

Therefore, Equation (21) reduces to

\[ f(\lambda) = \lambda^4 + \lambda^2 \left( 4 - \frac{1}{\mu_2} U_{x_2x_2} - U_{y_2y_2} \right) + \frac{1}{\mu_2^2} U_{x_2x_2} U_{y_2y_2} = 0 \]  

(21a)

**Case I:** Let \( \lambda_1^2 \) and \( \lambda_2^2 \) be the roots of equation (21a).

Sub case (i): If $\lambda_1^2$ and $\lambda_2^2$ both are real and one of them is positive, let $\lambda_1^2$ is a positive quantity, then square root of $\lambda_1^2$ must be real and of opposite sign. In this case characteristic roots will be a real number. So in this case $L_2$ is unstable. We may get similar result in the case of $\lambda_2^2$.

Sub case (ii): If $\lambda_1^2$ and $\lambda_2^2$ both are real and negative, let $\lambda_1^2$ is a negative quantity, then two roots of $\lambda_1^2$ be purely imaginary. Similarly two roots of $\lambda_2^2$ be purely imaginary. In this case $L_2$ should be stable. Again when $\lambda_1^2$ and $\lambda_2^2$ both are negative quantity then

$$\lambda_1^2 \lambda_2^2 = a \text{ positive quantity}$$

From equation (21a), we have obtained that

$$\lambda_1^2 \lambda_2^2 = \frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} = a \text{ negative quantity } \left( \cdot U_{x_{x_2}} > 0, \ U_{y_{y_2}} < 0 \right)$$

Since (21b) and (21c) contradict each other. So $L_2$ is unstable.

Now let $4 - \frac{1}{\mu_2} U_{x_{x_2}} - U_{y_{y_2}} = A_{24}$

Case II: If $A_{24} > 0$ i.e., $4 > \frac{1}{\mu_2} U_{x_{x_2}} + U_{y_{y_2}}$

Sub case (i): Let $U_{y_{y_2}} > 0$ then $\frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} > 0$ so, $f(\lambda) = \lambda^4 + A_{24} \lambda^2 + \frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} = 0$ has no change of signs and as such it has no positive real roots. $f(-\lambda) = \lambda^4 + A_{24} \lambda^2 + \frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} = 0$ also has no change of signs and as such it has no positive real roots i.e., $f(\lambda)$ has no negative real roots. So in this case we can say that all roots of $f(\lambda) = 0$ are imaginary.

Sub case (ii): If $U_{y_{y_2}} < 0$ then similar to Yadav’s case $\frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} < 0$.

Let $\frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} = -B_{24}, B_{24} > 0$, so $f(\lambda) = \lambda^4 + A_{24} \lambda^2 - B_{24} = 0$. Here $f(\lambda) = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 + A_{24} \lambda^2 - B_{24} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

Case III: If $A_{24} < 0$ i.e., $4 < \frac{1}{\mu_2} U_{x_{x_2}} + U_{y_{y_2}}$ then let $A_{24} = -C_{24}, C_{24} > 0$ then

$$f(\lambda) = \lambda^4 - C_{24} \lambda^2 + \frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} = 0$$

Sub case (i): Let $U_{y_{y_2}} > 0$, then $\frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} > 0$. Then, $f(\lambda) = 0$ has two changes of signs and as such it has at most two positive real roots. $f(-\lambda) = \lambda^4 - C_{24} \lambda^2 + \frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} = 0$ also has two change of signs and as such it has at most two positive real roots i.e., $f(\lambda)$ has at most two negative real roots.

Sub case (ii): If $U_{y_{y_2}} < 0$, then $\frac{1}{\mu_2} U_{x_{x_2}} U_{y_{y_2}} < 0$.
Let \( \frac{1}{\mu} U_{x_{24}} U_{y_{24}} = -D_{24}, D_{24} > 0 \). Here \( f(\lambda) = \lambda^4 - C_{24} \lambda^2 - D_{24} = 0 \) has only one change of sign and as such it has one positive real root. \( f(-\lambda) = \lambda^4 - C_{24} \lambda^2 - D_{24} = 0 \) has only one change of sign and as such it has one positive real root i.e., \( f(\lambda) \) has one negative real root. In both cases we see that \( f(\lambda) = 0 \) has one positive and one negative real root.

3. CONCLUSION
Hence, we get that the libration point \( L_2 \) is unstable.

### Table 1: For Stability of Collinear Equilibrium Solutions \( L_2 \)

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Figure 7

Figure 8

Figure 9

Figure 10

Figure 11

Figure 12
REFERENCES

[1] Z. Hoque and D. N. Garain: “Computation of $L_2$ in 2+2 body problem when perturbation effects act in coriolis and centrifugal forces, small primary is a radiating body and bigger primary is a triaxial rigid body” OIIRJ 2014; Vol. - IV, Issue - I, pp. 92-100.


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