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Characteristic Exponent of Collinear Libration Point in L_2 in Photo – gravitational Restricted Problem of 2+2 Bodies When Bigger Primary is a Triaxial Rigid Body Perturbed by Coriolis and Centrifugal Forces

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Abstract - *Stability of collinear libration point L_2 in photo – gravitational restricted problem of 2+2 bodies when bigger primary is a triaxial rigid body perturbed by coriolis and centrifugal forces has been studied in which it is found that L_2 is unstable.*

***Key Words*: Stability, Collinear libration point, Photo-gravitation, Triaxial rigid body, Coriolis force, Centrifugal force.**

1. INTRODUCTION

Equilibrium solutions of restricted problem of 2+2 bodies are derived by Whipple (1984) in which M_1 and M_2 are two point masses moving in the circular Keplerian orbit about their centre of mass. He assumed that $M_1 \geq M_2$. Two minor bodies m_1 and m_2 $(m_1, m_2 << M_2)$ move in the gravitational fields of primaries $(M_1 \text{ and } M_2)$. They attract each other but do not perturb the primaries. He showed the existence of fourteen equilibrium solutions. Six of these solutions are located about the collinear Lagrangian points of classical restricted problem of three bodies, eight solutions are found in the neighborhood of triangular Lagrangian points.

Sharma, Taqvi and Bhatnagar (2001) studied the existence and stability of libration points in the restricted three body problem when primaries are triaxial rigid body and source of radiation. They found five libration points, two triangular and three collinear they also observed that collinear points are unstable while triangular libration points are stable for the mass parameter $0 \leq \mu < \mu_{\text{crit}}$.

Garain and Chakraborty (2007) derived libration points and examined stability in Robe's three body restricted problem when second primary is a triaxial rigid body perturbed by coriolis and centrifugal forces. They found that the collinear libration points are deviated due to perturbation of centrifugal forces and triaxility of the second primary. Perturbation of coriolis force and triaxility character of the body play important role for finding the region of stability.

Hoque and Garain (2014) computed collinear libration point L_2. In the case of 2+2 body problem when perturbation effects act in coriolis and centrifugal forces, small primary is a radiating body and bigger primary as a triaxial rigid body.

2. EQUATIONS OF MOTION

Whipple's (1984) equation of motion of restricted problem of 2 + 2 bodies in synodic system be

$$\ddot{x}_i - 2\dot{y}_i = \frac{\partial T}{\partial x_i} \tag{1}$$

$$\ddot{y}_i + 2\dot{x}_i = \frac{\partial T}{\partial y_i} \tag{2}$$

$$\ddot{z}_i = \frac{\partial T}{\partial z_i}, \qquad (i =1, 2) \tag{3}$$

$$T=\sum_{i=1}^{2}\mu_i\left[\frac{(x_i^2+y_i^2)}{2}+\frac{(1-\mu)}{r_{1i}}+\frac{\mu}{r_{2i}}+\frac{\mu_{3-i}}{2r}\right]$$

$$\mu=\frac{M_2}{M_1+M_2},\ \mu_i=\frac{m_i}{M_1+M_2},\ (i=1,2),\qquad r_{1i}^2=(x_i-\mu)^2+y_i^2+z_i^2,\qquad r_{2i}^2=(x_i-\mu+1)^2+y_i^2+z_i^2$$

$$r^2=(x_1-x_2)^2+(y_1-y_{2)})^2+(z_1-z_2)^2$$

Here we consider M_2, a radiating body and M_1, a triaxial rigid body, we have also considered effect of perturbation in coriolis and centrifugal forces in this configuration.

Hence force function T reduces to U as follows:

$$U=\sum_{i=1}^{2}\mu_i\left[\frac{\beta(x_i^2+y_i^2)}{2}+\frac{(1-\mu)}{r_{1i}}+\frac{q\mu}{r_{2i}}+\frac{\mu_{3-i}}{2r}+\frac{(1-\mu)(2\sigma_1-\sigma_2)}{2r_{1i}^3}-\frac{3(1-\mu)(\sigma_1-\sigma_2)y_i^2}{2r_{1i}^5}\right] \quad (4)$$

Where, $q=1-\in,\ \beta=1+\in'$ and $i=1,\ 2$.

Equilibrium points of the system are those points where $\dot{x}_i=\dot{y}_i=\dot{z}_i=\frac{\partial U}{\partial x_i}=\frac{\partial U}{\partial y_i}=\frac{\partial U}{\partial z_i}=0,\ (i=1,2)$.

Thus we have,

$$\beta x_1-\frac{(1-\mu)(x_1-\mu)}{r_{11}^3}-\frac{q\mu(x_1-\mu+1)}{r_{21}^3}-\frac{\mu_2(x_1-x_2)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)}{2r_{11}^5}$$
$$+\frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1^2}{2r_{11}^7}=0 \quad (5)$$

$$\beta y_1-\frac{(1-\mu)y_1}{r_{11}^3}-\frac{q\mu y_1}{r_{21}^3}-\frac{\mu_2(y_1-y_2)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)y_1}{2r_{11}^5}$$
$$-\frac{3(1-\mu)(\sigma_1-\sigma_2)}{2}\left\{\frac{2y_1}{r_{11}^5}-\frac{5y_1^3}{r_{11}^7}\right\}=0 \quad (6)$$

$$-\frac{(1-\mu)z_1}{r_{11}^3}-\frac{q\mu z_1}{r_{21}^3}-\frac{\mu_2(z_1-z_2)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)z_1}{2r_{11}^5}+\frac{15(1-\mu)(\sigma_1-\sigma_2)y_1^2z_1}{2r_{11}^7}=0 \quad (7)$$

$$\beta x_2-\frac{(1-\mu)(x_2-\mu)}{r_{12}^3}-\frac{q\mu(x_2-\mu+1)}{r_{22}^3}-\frac{\mu_1(x_2-x_1)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_2-\mu)}{2r_{12}^5}$$
$$+\frac{15(1-\mu)(\sigma_1-\sigma_2)(x_2-\mu)y_2^2}{2r_{12}^7}=0 \quad (8)$$

$$\beta y_2-\frac{(1-\mu)y_2}{r_{12}^3}-\frac{q\mu y_2}{r_{22}^3}-\frac{\mu_1(y_2-y_1)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)y_2}{2r_{12}^5}$$
$$-\frac{3(1-\mu)(\sigma_1-\sigma_2)}{2}\left\{\frac{2y_2}{r_{12}^5}-\frac{5y_2^3}{r_{12}^7}\right\}=0 \quad (9)$$

$$-\frac{(1-\mu)z_2}{r_{12}^3}-\frac{q\mu z_2}{r_{22}^3}-\frac{\mu_1(z_2-z_1)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)z_2}{2r_{12}^5}+\frac{15(1-\mu)(\sigma_1-\sigma_2)y_2^2z_2}{2r_{12}^7}=0 \quad (10)$$

From equations (7) and (10), we have $z_1=z_2=0$.

By inspection, it can be seen that equations (6) and (9) are satisfied when $y_1=y_2=0$.

Now we have to determine x_1 and x_2 such that the following simplified forms of equations (5) and (8) are satisfied.

$$\therefore \beta x_1 - \frac{(1-\mu)(x_1-\mu)}{|x_1-\mu|^3} - \frac{q\mu(x_1-\mu+1)}{|x_1-\mu+1|^3} - \frac{\mu_2(x_1-x_2)}{|x_1-x_2|^3}$$

$$-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)}{2|x_1-\mu|^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1^2}{2|x_1-\mu|^7} = 0 \quad (11)$$

And $$\beta x_2 - \frac{(1-\mu)(x_2-\mu)}{|x_2-\mu|^3} - \frac{q\mu(x_2-\mu+1)}{|x_2-\mu+1|^3} - \frac{\mu_2(x_2-x_1)}{|x_2-x_1|^3}$$

$$-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_2-\mu)}{2|x_2-\mu|^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)(x_2-\mu)y_2^2}{2|x_2-\mu|^7} = 0 \quad (12)$$

The solution of equations (11) and (12) can be obtained with the help of power series.

Let $\in_i = \dfrac{\mu_i}{(\mu_1+\mu_2)^{\frac{2}{3}}}$, $(i=1,2)$ (13)

$$\therefore \in_1 = \frac{\mu_1}{(\mu_1+\mu_2)^{\frac{2}{3}}} \text{ and } \in_2 = \frac{\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}} \quad (14)$$

$$\therefore \mu_2 \in_1 = \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}} = \mu_1 \in_2 = k(say) \quad (15)$$

Let $x_1 = L_i' + \sum_{j=1}^{n} a_{1j} \in_2^j$, for $i = 1,\ 2,\ 3$ (16)

Where $L_1',\ L_2',\ L_3'$ equilibrium points be in photo-gravitational restricted problem of three bodies when bigger primary is a triaxial rigid body perturbed by coriolis and centrifugal forces and x_1 be the x coordinate of first small body.

$$x_2 = L_i' + \sum_{j=1}^{n} a_{2j} \in_1^j \text{ for } i = 1, 2, 3 \quad (17)$$

Similar to Whipple,

$$x_1 = L_i' + \frac{(\pm 1)}{(\Omega_{xx}^{\circ})^{\frac{1}{3}}} \frac{\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}} \text{ where } i = 1, 2, 3 \quad (18)$$

And

$$x_2 = L_i' - \frac{(\pm 1)}{(\Omega_{xx}^{\circ})^{\frac{1}{3}}} \frac{\mu_1}{(\mu_1+\mu_2)^{\frac{2}{3}}} \text{ where } i = 1, 2, 3 \quad (19)$$

Hoque and Garain (2014) obtained two values $(x_1, 0, 0)$ and $(x_2, 0, 0)$ of L_2 in which $x_1 = a_{21} - b_{21} \in + c_{21} \in' - 2d_{21}\sigma_1 + d_{21}\sigma_2$
and
$x_2 = a_{22} - b_{22} \in + c_{22} \in' - 2d_{22}\sigma_1 + d_{22}\sigma_2$

Where $a_{21} = \mu - 1 + a_2 + \dfrac{(\pm 1)\mu_2}{A_2^{\frac{1}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $b_{21} = b_2 - \dfrac{(\pm 1)\mu_2 B_2}{3A_2^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

$$a_{22} = \mu - 1 + a_2 - \frac{(\pm 1)\mu_1}{A_2^{\frac{1}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}, \quad b_{22} = b_2 + \frac{(\pm 1)\mu_1 B_2}{3A_2^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$$

$$c_{22} = c_2 + \frac{(\pm 1)\mu_1 C_2}{3A_2^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}} \text{ and } d_{22} = d_2 - \frac{(\pm 1)\mu_1 D_2}{3A_2^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$$

$$c_{21} = c_2 - \frac{(\pm 1)\mu_2 C_2}{3A_2^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}, \quad d_{21} = d_2 + \frac{(\pm 1)\mu_2 D_2}{3A_2^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$$

They obtained the particular values of x_1 and x_2 for different values of μ, μ_1 and μ_2.

Our characteristic equation corresponding to the point $(x_1, 0, 0)$ is

$$f(\lambda) = \lambda^4 + \lambda^2\left(4 - \frac{U_{x_1x_1}}{\mu_1} - U_{y_1y_1}\right) + \frac{1}{\mu_1^2}\left(U_{x_1x_1}U_{y_1y_1} - U_{x_1y_1}^2\right) = 0 \tag{20}$$

$$\frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{r_{11}^3} + \frac{2q\mu}{r_{21}^3} + \frac{2\mu_2}{r^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{r_{11}^5} - \frac{45(1-\mu)(\sigma_1-\sigma_2)y_1^2}{r_{11}^7}\right]$$

$$\frac{U_{y_1y_1}}{\mu_1} = \left[\beta - \frac{(1-\mu)}{r_{11}^3} - \frac{q\mu}{r_{21}^3} - \frac{\mu_2}{r^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2r_{11}^5} + \frac{3(1-\mu)y_1^2}{r_{11}^5} + \frac{3q\mu y_1^2}{r_{21}^5}\right.$$
$$\left. + \frac{3\mu_2(y_1-y_2)^2}{r^5} + \frac{15(1-\mu)(7\sigma_1-6\sigma_2)y_1^2}{2r_{11}^7} - \frac{105(1-\mu)(\sigma_1-\sigma_2)y_1^4}{2r_{11}^9}\right]$$

$$U_{x_1y_1} = \mu_1\left[\frac{3(1-\mu)(x_1-\mu)y_1}{r_{11}^5} + \frac{3q\mu(x_1-\mu+1)y_1}{r_{21}^5} + \frac{3\mu_2(x_1-x_2)(y_1-y_2)}{r^5}\right.$$
$$\left. + \frac{15(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)y_1}{2r_{11}^7} + \frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1}{r_{11}^7}\right]$$

In this case, $y_1 = 0$ and $y_2 = 0$.

Therefore the above equations reduce to

$$\frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{(x_1-\mu)^3} + \frac{2q\mu}{(x_1-\mu+1)^3} + \frac{2\mu_2}{(x_1-x_2)^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(x_1-\mu)^5}\right]$$

$$\frac{U_{x_1y_1}}{\mu_1} = 0$$

And

$$\frac{U_{y_1y_1}}{\mu_1} = \left[\beta - \frac{(1-\mu)}{(x_1-\mu)^3} - \frac{q\mu}{(x_1-\mu+1)^3} - \frac{\mu_2}{(x_1-x_2)^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2(x_1-\mu)^5}\right]$$

For the collinear equilibrium solutions the partial derivatives contained in equation (20) reduce to

$$\frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{|x_1-\mu|^3} + \frac{2q\mu}{|x_1-\mu+1|^3} + \frac{2\mu_2}{|x_1-x_2|^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{|x_1-\mu|^5}\right]$$

$$\frac{U_{x_1y_1}}{\mu_1}=0$$

$$\frac{U_{y_1y_1}}{\mu_1}=\left[(1+\in')-\frac{(1-\mu)}{|x_1-\mu|^3}-\frac{(1-\in)\mu}{|x_1-\mu+1|^3}-\frac{\mu_2}{|x_1-x_2|^3}-\frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2|x_1-\mu|^5}\right]$$

Let $\frac{U_{y_1y_1}}{\mu_1}=A_{21}-B_{21}\in+C_{21}\in'-D_{21}\sigma_1+E_{21}\sigma_2$

$$x_1=a_{21}-b_{21}\in+c_{21}\in'-2d_{21}\sigma_1+d_{21}\sigma_2$$

and

$$x_2=a_{22}-b_{22}\in+c_{22}\in'-2d_{22}\sigma_1+d_{22}\sigma_2$$

$$\Rightarrow\frac{U_{y_1y_1}}{\mu_1}=A_{21}-B_{21}\in+C_{21}\in'-D_{21}\sigma_1+E_{21}\sigma_2$$

Where,

$$A_{21}=1-\frac{(1-\mu)}{|a_{21}-\mu|^3}-\frac{\mu}{|a_{21}-\mu+1|^3}-\frac{\mu_2}{|a_{21}-a_{22}|^3},$$

$$B_{21}=\frac{3(1-\mu)b_{21}}{|a_{21}-\mu|^4}-\frac{\mu}{|a_{21}-\mu+1|^3}+\frac{3\mu b_{21}}{|a_{21}-\mu+1|^4}+\frac{3\mu_2(b_{21}-b_{22})}{|a_{21}-a_{22}|^4},$$

$$C_{21}=1+\frac{3(1-\mu)c_{21}}{|a_{21}-\mu|^4}+\frac{3\mu c_{21}}{|a_{21}-\mu+1|^4}+\frac{3\mu_2(c_{21}-c_{22})}{|a_{21}-a_{22}|^4},$$

$$D_{21}=\frac{6(1-\mu)d_{21}}{|a_{21}-\mu|^4}+\frac{6\mu d_{21}}{|a_{21}-\mu+1|^4}+\frac{6\mu_2(d_{21}-d_{22})}{|a_{21}-a_{22}|^4}+\frac{6(1-\mu)}{|a_{21}-\mu|^5}$$

and

$$E_{21}=\frac{3(1-\mu)d_{21}}{|a_{21}-\mu|^4}+\frac{3\mu d_{21}}{|a_{21}-\mu+1|^4}+\frac{3\mu_2(d_{21}-d_{22})}{|a_{21}-a_{22}|^4}+\frac{9(1-\mu)}{2|a_{21}-\mu|^5}$$

Here we see that $U_{x_1x_1}>0$ and $U_{x_1y_1}=0$. Therefore equation (20) reduces to

$$f(\lambda)=\lambda^4+\lambda^2\left(4-\frac{1}{\mu_1}U_{x_1x_1}-U_{y_1y_1}\right)+\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=0 \qquad (20a)$$

Case I: Let λ_1^2 and λ_2^2 be the roots of equation (20a).

Sub case (i): If λ_1^2 and λ_2^2 both are real and one of them is positive, let λ_1^2 is a positive quantity, then square root of λ_1^2 must be real and of opposite sign. In this case characteristic roots will be a real number. So in this case L_2 is unstable. We may get similar result in the case of λ_2^2.

Sub case (ii) If λ_1^2 and λ_2^2 both are real and negative, let λ_1^2 is a negative quantity, then two roots of λ_1^2 be purely imaginary. Similarly two roots of λ_2^2 be purely imaginary. In this case L_2 should be stable. Again when λ_1^2 and λ_2^2 both are negative quantity then

$\lambda_1^2\lambda_2^2$ = a positive quantity (20b)

From equation (20a), we have obtained that

$\lambda_1^2\lambda_2^2 = \dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}$ = a negative quantity $\left(\because U_{x_1x_1} > 0,\ U_{y_1y_1} < 0\right)$ (20c)

Since (20b) and (20c) contradict each other. So L_2 is unstable.

Now let $4 - \dfrac{U_{x_1x_1}}{\mu_1} - U_{y_1y_1} = A_{23}$

Case II: If $A_{23} > 0$ i.e., $4 > \dfrac{1}{\mu_1}U_{x_1x_1} + U_{y_1y_1}$ then $f(\lambda) = \lambda^4 + A_{23}\lambda^2 + \dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$.

Sub case (i): It is clear that $U_{x_1x_1} > 0$ and let $U_{y_1y_1} > 0$ then $\dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} > 0$ so, $f(\lambda) = 0$ has no change of signs and as such it has no positive real roots. $f(-\lambda) = \lambda^4 + A_{23}\lambda^2 + \dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$ also has no change of signs and as such it has no positive real roots i.e., $f(\lambda)$ has no negative real roots. So in this case we can say that all roots of $f(\lambda) = 0$ are imaginary.

Sub case (ii): If $U_{y_1y_1} < 0$ then similar to Yadav's case $\dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} < 0$.

Let $\dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = -B_{23}, B_{23} > 0$, so, $f(\lambda) = \lambda^4 + A_{23}\lambda^2 - B_{23} = 0$. Here $f(\lambda) = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 + A_{23}\lambda^2 - B_{23} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

Case III: If $A_{23} < 0$ i.e., $4 < \dfrac{1}{\mu_1}U_{x_1x_1} + U_{y_1y_1}$ then let $A_{23} = -C_{23}, C_{23} > 0$ then

$$f(\lambda) = \lambda^4 - C_{23}\lambda^2 + \frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$$

Sub case (i): Let $U_{y_1y_1} > 0$, then $\dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} > 0$.

Then, $f(\lambda) = 0$ has two changes of signs and as such it has at most two positive real roots. $f(-\lambda) = \lambda^4 - C_{23}\lambda^2 + \dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = 0$ also has two change of signs and as such it has at most two positive real roots i.e., $f(\lambda)$ has at most two negative real roots.

Sub case (ii): If $U_{y_1y_1} < 0$, then $\dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} < 0$.

Let $\dfrac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1} = -D_{23}, \mathrm{D}_{23} > 0$. Here $f(\lambda) = \lambda^4 - C_{23}\lambda^2 - D_{23} = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 - C_{23}\lambda^2 - D_{23} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

In both the cases we see that $f(\lambda) = 0$ has one positive and one negative real root. So the libration point L_2 is unstable.

The characteristic equation corresponding to the point $(x_2, 0, 0)$ be

$$f(\lambda)=\lambda^4+\lambda^2\left(4-\frac{1}{\mu_2}U_{x_2x_2}-U_{y_2y_2}\right)+\frac{1}{\mu_2^2}\left(U_{x_2x_2}U_{y_2y_2}-U_{x_2y_2}^2\right)=0 \tag{21}$$

$$\frac{U_{x_2x_2}}{\mu_2}=\left[\beta+\frac{2(1-\mu)}{(x_2-\mu)^3}+\frac{2q\mu}{(x_2-\mu+1)^3}+\frac{2\mu_1}{(x_1-x_2)^3}+\frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(x_2-\mu)^5}\right]$$

$$\frac{U_{y_2y_2}}{\mu_2}=\left[\beta-\frac{(1-\mu)}{(x_2-\mu)^3}-\frac{q\mu}{(x_2-\mu+1)^3}-\frac{\mu_1}{(x_1-x_2)^3}-\frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2(x_2-\mu)^5}\right]$$

$$\frac{U_{x_2y_2}}{\mu_2}=0$$

$$\frac{U_{x_2x_2}}{\mu_2}=\left[\beta+\frac{2(1-\mu)}{|x_2-\mu|^3}+\frac{2q\mu}{|x_2-\mu+1|^3}+\frac{2\mu_1}{|x_1-x_2|^3}+\frac{6(1-\mu)(2\sigma_1-\sigma_2)}{|x_2-\mu|^5}\right]$$

$$\frac{U_{x_2y_2}}{\mu_2}=0$$

$$\frac{U_{x_2y_2}}{\mu_2}=\left[(1+\in')-\frac{(1-\mu)}{|x_2-\mu|^3}-\frac{(1-\in)\mu}{|x_2-\mu+1|^3}-\frac{\mu_1}{|x_1-x_2|^3}-\frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2|x_2-\mu|^5}\right]$$

Let $\dfrac{U_{y_2y_2}}{\mu_2}=A_{22}-B_{22}\in+C_{22}\in'-D_{22}\sigma_1+E_{22}\sigma_2$ $x_1=a_{21}-b_{21}\in+c_{21}\in'-2d_{21}\sigma_1+d_{21}\sigma_2$

and

$x_2=a_{22}-b_{22}\in+c_{22}\in'-2d_{22}\sigma_1+d_{22}\sigma_2$

Where,

$$A_{22}=1-\frac{(1-\mu)}{|a_{22}-\mu|^3}-\frac{\mu}{|a_{22}-\mu+1|^3}-\frac{\mu_1}{|a_{22}-a_{21}|^3},$$

$$B_{22}=\frac{3(1-\mu)b_{22}}{|a_{22}-\mu|^4}-\frac{\mu}{|a_{22}-\mu+1|^3}+\frac{3\mu b_{22}}{|a_{22}-\mu+1|^4}+\frac{3\mu_1(b_{22}-b_{21})}{|a_{22}-a_{21}|^4},$$

$$C_{22}=1+\frac{3(1-\mu)c_{22}}{|a_{22}-\mu|^4}+\frac{3\mu c_{22}}{|a_{22}-\mu+1|^4}+\frac{3\mu_1(c_{22}-c_{21})}{|a_{22}-a_{21}|^4},$$

$$D_{22}=\frac{6(1-\mu)d_{22}}{|a_{22}-\mu|^4}+\frac{6\mu d_{22}}{|a_{22}-\mu+1|^4}+\frac{6\mu_1(d_{22}-d_{21})}{|a_{22}-a_{21}|^4}+\frac{6(1-\mu)}{|a_{22}-\mu|^5}$$

and

$$E_{22}=\frac{3(1-\mu)d_{22}}{|a_{22}-\mu|^4}+\frac{3\mu d_{22}}{|a_{22}-\mu+1|^4}+\frac{3\mu_1(d_{22}-d_{21})}{|a_{22}-a_{21}|^4}+\frac{9(1-\mu)}{2|a_{22}-\mu|^5}$$

Here we see that $U_{x_2x_2}>0$ and $U_{x_2y_2}=0$

Therefore, Equation (21) reduces to

$$f(\lambda)=\lambda^4+\lambda^2\left(4-\frac{1}{\mu_2}U_{x_2x_2}-U_{y_2y_2}\right)+\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2}=0 \tag{21a}$$

Case I: Let λ_1^2 and λ_2^2 be the roots of equation (21a).

Sub case (i): If λ_1^2 and λ_2^2 both are real and one of them is positive, let λ_1^2 is a positive quantity, then square root of λ_1^2 must be real and of opposite sign. In this case characteristic roots will be a real number. So in this case L_2 is unstable. We may get similar result in the case of λ_2^2.

Sub case (ii): If λ_1^2 and λ_2^2 both are real and negative, let λ_1^2 is a negative quantity, then two roots of λ_1^2 be purely imaginary. Similarly two roots of λ_2^2 be purely imaginary. In this case L_2 should be stable. Again when λ_1^2 and λ_2^2 both are negative quantity then

$\lambda_1^2\lambda_2^2$ = a positive quantity (21b)

From equation (21a), we have obtained that

$\lambda_1^2\lambda_2^2 = \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2}$ = a negative quantity $\left(\because U_{x_2x_2} > 0,\ \ U_{y_2y_2} < 0\right)$ (21c)

Since (21b) and (21c) contradict each other. So L_2 is unstable.

Now let $4 - \frac{1}{\mu_2}U_{x_2x_2} - U_{y_2y_2} = A_{24}$

Case II: If $A_{24} > 0$ i.e., $4 > \frac{1}{\mu_2}U_{x_2x_2} + U_{y_2y_2}$.

Sub case (i): Let $U_{y_2y_2} > 0$ then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} > 0$ so, $f(\lambda) = \lambda^4 + A_{24}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ has no change of signs and as such it has no positive real roots. $f(-\lambda) = \lambda^4 + A_{24}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ also has no change of signs and as such it has no positive real roots i.e., $f(\lambda)$ has no negative real roots. So in this case we can say that all roots of $f(\lambda) = 0$ are imaginary.

Sub case (ii): If $U_{y_2y_2} < 0$ then similar to Yadav's case $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} < 0$.

Let $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = -B_{24}, B_{24} > 0,$ so $f(\lambda) = \lambda^4 + A_{24}\lambda^2 - B_{24} = 0$. Here $f(\lambda) = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 + A_{24}\lambda^2 - B_{24} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

Case III: If $A_{24} < 0$ i.e., $4 < \frac{1}{\mu_2}U_{x_2x_2} + U_{y_2y_2}$ then let $A_{24} = -C_{24}, C_{24} > 0$ then

$f(\lambda) = \lambda^4 - C_{24}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$

Sub case (i): Let $U_{y_2y_2} > 0$, then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} > 0$. Then, $f(\lambda) = 0$ has two changes of signs and as such it has at most two positive real roots. $f(-\lambda) = \lambda^4 - C_{24}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ also has two change of signs and as such it has at most two positive real roots i.e., $f(\lambda)$ has at most two negative real roots.

Sub case (ii): If $U_{y_2y_2} < 0$, then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} < 0$.

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Let $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = -D_{24},\ D_{24} > 0.$ Here $f(\lambda) = \lambda^4 - C_{24}\lambda^2 - D_{24} = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 - C_{24}\lambda^2 - D_{24} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root. In both cases we see that $f(\lambda) = 0$ has one positive and one negative real root.

3. CONCLUSION

Hence, we get that the libration point L_2 is unstable.

Table 1: For Stability of Collinear Equilibrium Solutions L_2

$$\left(\mu_1 = \mu_2 = 10^{-10},\ \frac{1}{\mu_1}U_{y_1y_1} = A_{21} - B_{21}\in + C_{21}\in' - D_{21}\sigma_1 + E_{21}\sigma_2\right)$$

μ	A_{21}	B_{21}	C_{21}	D_{21}	E_{21}
0.00001	-7.6016500325775700	-0.0597841351601680	126.3470985203830000	420.6183313549640000	211.9268914158130000
0.00002	-7.6688038183727900	-0.0637540777253726	99.4401534217220000	340.3880774630580000	171.8436125664220000
0.00003	-7.7128339342020800	-0.0679209208165934	86.2856743432114000	301.2553225133020000	152.2999600130830000
0.00004	-7.7468462123541200	-0.0718131592180664	77.9494193125515000	276.5057714426010000	139.9435044387700000
0.00005	-7.7750916306424000	-0.0754223356049816	71.9996616375487000	258.8738209899750000	131.1431587394000000
0.00006	-7.7995290033286800	-0.0787866192759827	67.4490464863296000	245.4112918774030000	124.4256761723530000
0.00007	-7.8212365147090800	-0.0819438560016496	63.8067037722911000	234.6533182898820000	119.0591087345410000
0.00008	-7.8408765677507900	-0.0849253428684929	60.7959253279933000	225.7745818649720000	114.6311050901930000
0.00009	-7.8588879316127400	-0.0877561317394450	58.2466539039960000	218.2681201009630000	110.8883932026980000
0.0001	-7.8755776596932200	-0.0904562771146028	56.0475441350395000	211.8021576730380000	107.6652348959410000
0.0002	-8.0011422517553600	-0.1127776579606220	43.3128180892842000	174.5959917786500000	89.1379142724235000
0.0003	-8.0899277102024900	-0.1303088966309170	37.0846450439462000	156.6251273730950000	80.2075302045291000
0.0004	-8.1614059079372000	-0.1452900263286190	33.1323115550649000	145.3463645569560000	74.6131132336267000
0.0005	-8.2224372531703400	-0.1586339294667430	30.3065125747827000	137.3649477622550000	70.6611726961538000
0.0006	-8.2763427688745200	-0.1708167258938770	28.1410253479885000	131.3081261335170000	67.6672530507577000
0.0007	-8.3250149310807300	-0.1821231256439350	26.4041839704806000	126.4957411785890000	65.2923837662741000
0.0008	-8.3696474943837900	-0.1927390562619860	24.9654350249212000	122.5455256704630000	63.3461374352981000
0.0009	-8.4110480133641200	-0.2027936815871070	23.7445568602571000	119.2231860210820000	61.7118486161199000
0.001	-8.4497915196701600	-0.2123809497743410	22.6890277452508000	116.3757037229750000	60.3133528997951000
0.002	-8.7516361649384800	-0.2928045204073460	16.5092315776725000	100.3424253887260000	52.4959016878852000
0.003	-8.9746086488896400	-0.3586631487532160	13.4122359438805000	92.9237849401226000	48.9359789790892000
0.004	-9.1591335250948000	-0.4173650964222890	11.3984098379392000	88.4443649134747000	46.8209667406454000
0.005	-9.3199927153586600	-0.4717183801436470	9.9230284350873300	85.3887432678844000	45.4024415579428000
0.006	-9.4644991879494300	-0.5231385655610340	8.7644941132045100	83.1510074179213000	44.3820763939948000
0.007	-9.5968844851274400	-0.5724544448953290	7.8123852037624600	81.4340144918736000	43.6139933411392000
0.008	-9.7198493908659000	-0.6201992315084560	7.0043016490238100	80.0723677348681000	43.0172208759002000
0.009	-9.8352384211161900	-0.6667391623404730	6.3017952824132400	78.9656269654117000	42.5427227512870000
0.01	-9.9443755352937900	-0.7123382276130470	5.6796258546001600	78.0487775008641000	42.1588441824431000
0.02	-10.8333731754558000	-1.1449869618632000	1.5950495245623200	73.6256789166793000	40.5465173363312000
0.03	-11.5371618142511000	-1.5723214918754300	-0.9834311830190710	72.1739583719075000	40.2733792392092000
0.04	-12.1543482474494000	-2.0151205198888500	-3.0492007679675500	71.5148158633937000	40.3149434691476000
0.05	-12.7222975497245000	-2.4822792271220000	-4.8819304964756400	71.1139337273100000	40.4286813420622000
0.06	-13.2602598611071000	-2.9794526628503700	-6.6002751680474000	70.7731598730914000	40.5280431787326000
0.07	-13.7799823058041000	-3.5111782275255800	-8.2670196076917400	70.4012981781677000	40.5748663829276000
0.08	-14.2895103995202000	-4.0816477591887100	-9.9208602442999200	69.9497072689730000	40.5497057856924000
0.09	-14.7948632427140000	-4.6950685760537200	-11.5887087016854000	69.3890613185627000	40.4413333851228000
0.1	-15.3008808679469000	-5.3558722304386800	-13.2912972041858000	68.6994813805504000	40.2423689982937000
0.2	-21.2175028647762000	-15.9757913762007000	-35.8691122892482000	51.6335683233382000	32.0491236066294000
0.3	-32.0904001555764000	-44.4002309686469000	-87.5811858691160000	-3.1857147038466600	3.5784188766858600
0.4	-62.3242233419039000	-157.9335404012020000	-277.7744483295110000	-232.8113628769960000	-112.9350793345430000
0.5	-233.0062837991820000	-1284.3495085767800000	-2028.2617803693100000	-2715.8703224220200000	-1356.1117484223600000

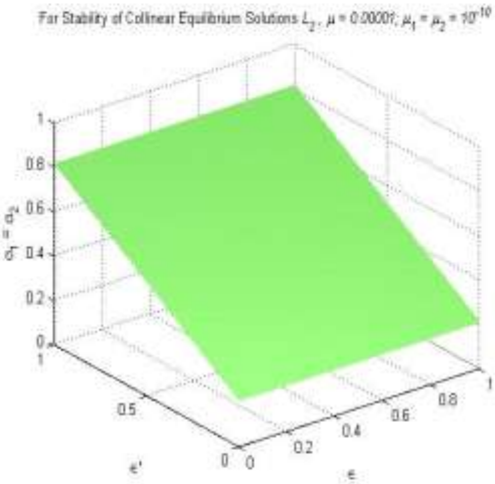
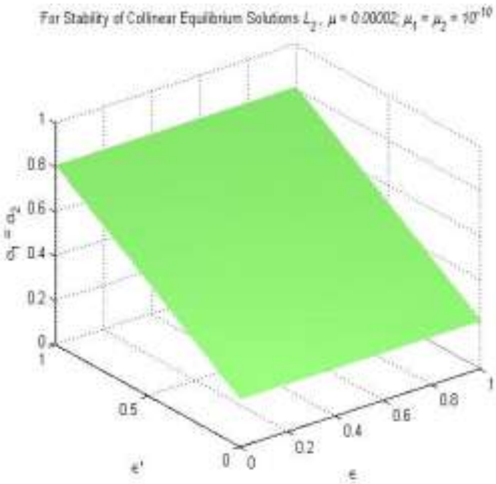


Figure 1



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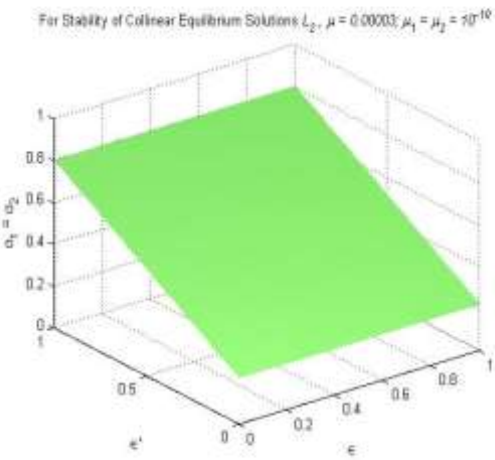

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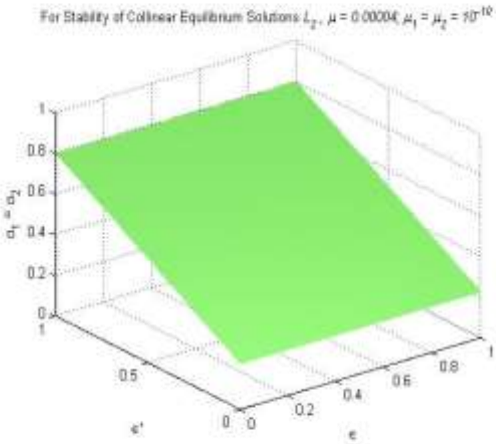

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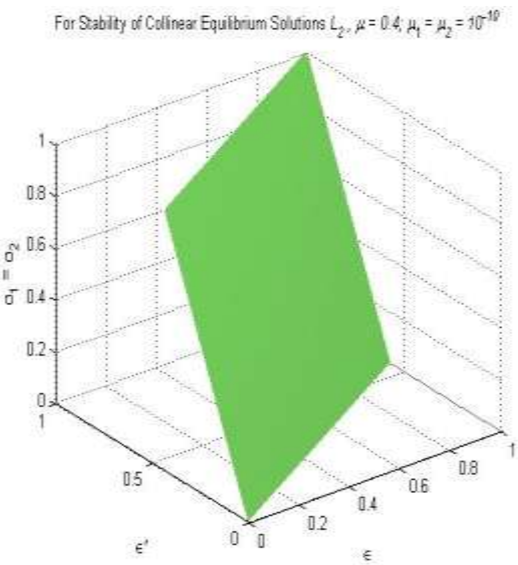

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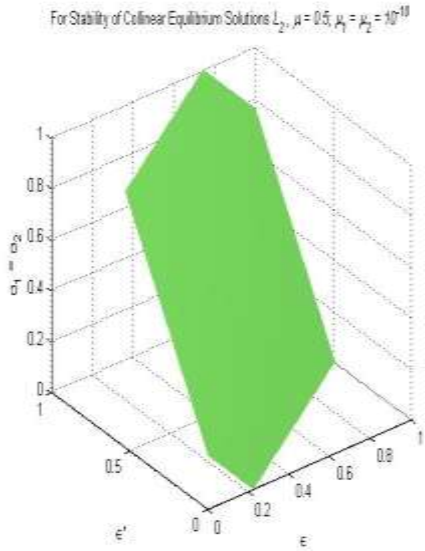

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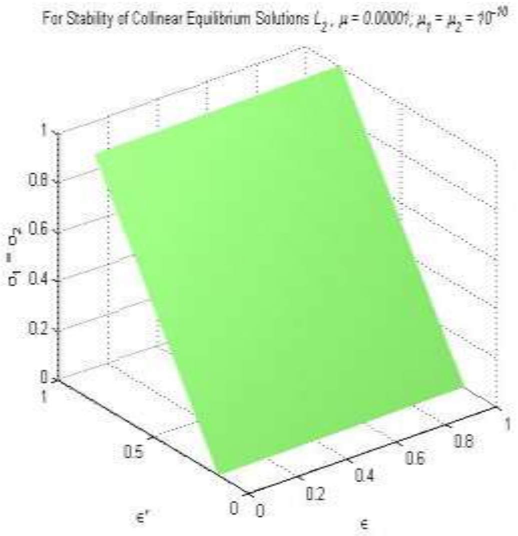

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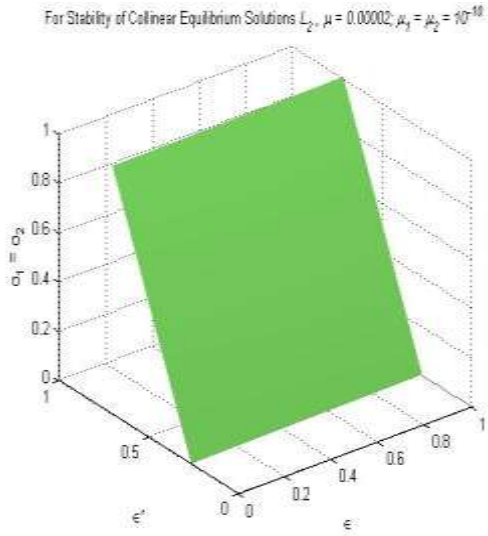

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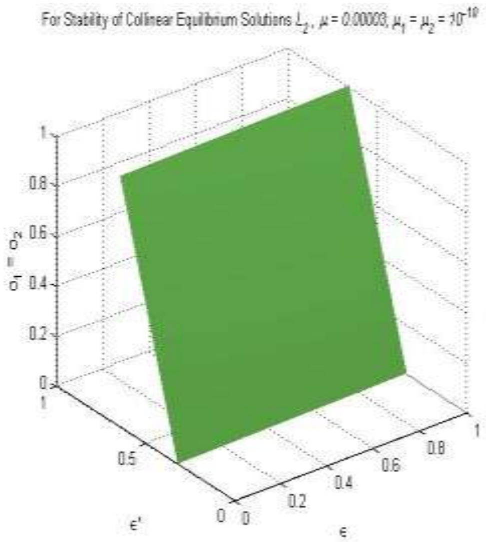

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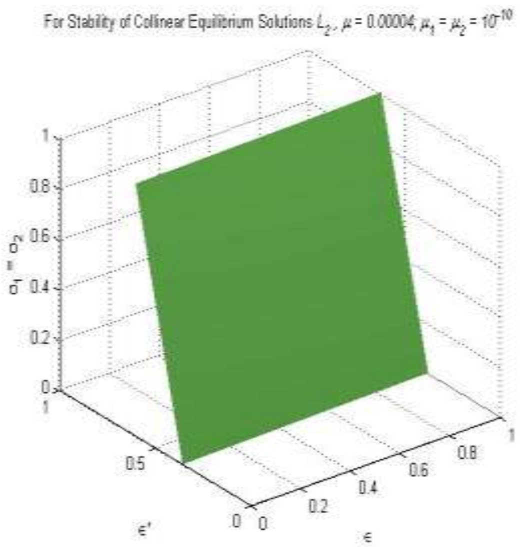
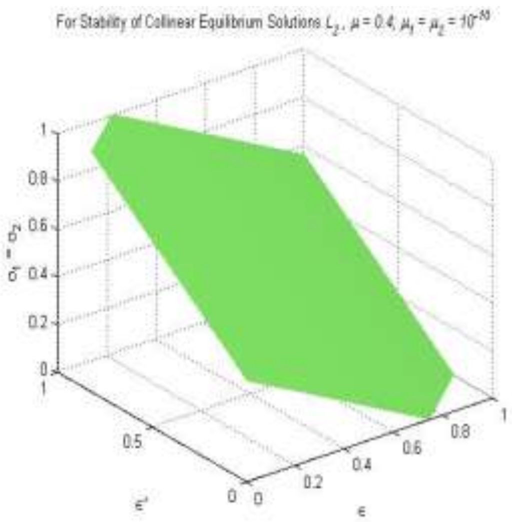


Figure 10



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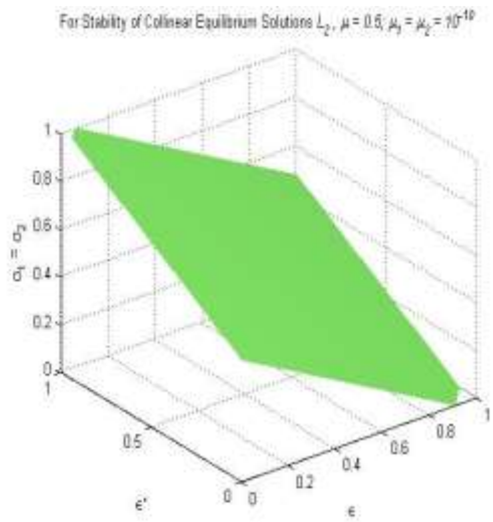

Figure 12

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