INTEGRAL SOLUTIONS OF THE SEXTIC DIOPHANTINE EQUATION WITH FIVE UNKNOWNS $x^3 - y^3 = 7(z-w)R^5$

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***Abstract** - The Non-homogeneous Diophantine equation in five unknowns x, y, z, w and R is given by $x^3 - y^3 = 7(z-w)R^5$ is analyzed for its patterns of non- zero integral solutions and a few interesting relations between the solutions and special polygonal numbers are presented.*

Keywords: Non-Homogeneous, integer solutions, polygonal number and Pronic number, Mersenne number.

1. INTRODUCTION

The Diophantine equations offer an unlimited field of research because of their variety [1- 4]. The solutions of Diophantine equations of higher degree are of greater importance in its applications in network securities for computer science and engineering. In particular one may refer [5-19] for finding integral solutions. In this paper i have made an attempt to discuss the integral solutions of a Diophantine equation of degree six in five unknowns in different patterns and some properties involving special numbers.

1.1 Notations Used:

1. $t_{m,n}$ =Polygonal number of rank 'n' with sides' m
2. Mer_n= Mersenne number
3. Pr_n= Pronic number of rank 'n'
4. g_n =Gnomonic number
5. $car\ I_n$=Carol number

2. METHOD OF ANALYSIS

Consider the non-homogeneous Diophantine equation in five unknowns solved for non-zero distinct integer solutions as given below

$x^3 - y^3 = 7(z-w)R^5$ (1)

Use the linear transformation $x = u + v$

$, y = u - v , z = 2u + v \ and \ w = 2u - v$ (2)

Thus (1) becomes $(v^2 + 3u^2) = 7R^5$ (3)

Assume that $R = b^2 + 3a^2$ (4)

Where a and b are non-zero integers

SOLUTION PATTERN-1

Write 7= $(2 + i\sqrt{3})(2 - i\sqrt{3})$ (5)

Using (4) and (5) in (3) and applying the method of factorization,

$$(v + i\sqrt{3}\,u) = (2 + i\sqrt{3})(b + i\sqrt{3}\,a)^5$$

Equating the real and imaginary parts,

$$u = u(a,b) = 10b^4a - 60b^2a^3 + 18a^5 + b^5 - 30b^3a^2 + 45ba^4$$

$$v = v(a,b) = -15b^4a + 90b^2a^3 - 27a^5 + 2b^5 - 60b^3a^2 + 90ba^4 \quad (6)$$

Substituting u and v from (6) in (2) and from (4), the integral solution is as given below

$$x = x(a,b) = -5b^4a + 30b^2a^3 - 9a^5 + 3b^5 - 90b^3a^2 + 135ba^4$$

$$y = y(a,b) = 25b^4a - 150b^2a^3 + 45a^5 - b^5 + 30b^3a^2 - 45ba^4$$

$$z = z(a,b) = 5b^4a - 30b^2a^3 + 6a^5 + 4b^5 - 120b^3a^2 + 180ba^4$$

$w = w(a,b) = 35b^4a - 210b^2a^3 + 63a^5$ and
$R = R(a,b) = b^2 + 3a^2$

PROPERTIES:

1. $x(A,A) - y(A,A) + z(A,A) - w(A,A) = 0(mod317)$
2. $R(a,a)\ is\ a\ perfect\ square.$
3. $x(A,A) + y(A,A) + z(A,A) + w(A,A) \equiv 0(mod99)$

4. $y(A,A)$ *is a nasty number*.

5. $x(2^n,2^n)+y(2^n,2^n)+z(2^n,2^n)+w(2^n,2^n)+\ (99Mer_{5n})+99=0$

6. $R(2^n,2^n)-4Pr_{2^n}+4(2^n)=0$

SOLUTION PATTERN-2

Write (3) as $(v^2+3u^2)=7*R^5$ (7)

Now write $7=\left(\frac{(2+i\sqrt{3})^3(2-i\sqrt{3})^3}{49}\right)$ (8)

Using (4), (8) in (7) and by the method of factorization, we get

$$\left(v+i\sqrt{3}\,u\right)=\frac{\left(2+i\sqrt{3}\right)^3}{7}\left(b+i\sqrt{3}\,a\right)^5$$

Equating the real and imaginary parts,

$$u=u(a,b)=\frac{1}{7}(-50b^4a+300b^2a^3-90a^5+9b^5-270b^3a^2+405ba^4)$$

$$v=v(a,b)=\frac{1}{7}(-135b^4a+810b^2a^3-243a^5-10b^5+300b^3a^2-450ba^4) \quad (9)$$

Substituting u and v from (9) in (2) and from (4), we get

$x=x(a,b)$

$$=\frac{1}{7}(-185b^4a+1110b^2a^3-333a^5-b^5+30b^3a^2-45ba^4)$$

$y=y(a,b)$

$$=\frac{1}{7}(85b^4a-510b^2a^3+153a^5+19b^5-570b^3a^2+855ba^4)$$

$z=z(a,b)$

$$=\frac{1}{7}(-235b^4a+1410b^2a^3-423a^5+8b^5+30b^3a^2+360ba^4)$$

$w=w(a,b)$

$$=\frac{1}{7}(35b^4a-210b^2a^3+63a^5+28b^5-840b^3a^2+1260ba^4)$$

$R=R(a,\ b)=b^2+3a^2$

For the integral solution replace a by 7A and b by 7B,

$x=x(A,B)$

$$=7^4(-185B^4A+1110B^2A^3-333A^5-B^5+30B^3A^2-45BA^4)$$

$y=y(A,B)$

$$=7^4(85B^4A-510B^2A^3+153A^5+19B^5-570B^3A^2+855BA^4)$$

$z=z(A,B)$

$$=7^4(-235B^4A+1410B^2A^3-423A^5+8B^5+30B^3A^2+360BA^4)$$

$w=w(A,B)$

$$=7^4(35B^4A-210B^2A^3+63A^5+28B^5-840B^3A^2+1260BA^4)$$

$R=R(A,\ B)=49(B^2+3A^2)$

PROPERTIES:

1. $x(A,A)+y(A,A)+z(A,A)+w(A,A)\equiv 0(mod14)$

2. $R(a,a)-t_{4,14a}=0$

3. $x(a^2,a^2)$ is a perfect square.

4. $x(A,A)-18y(A,A)=0$

5. $z(2^n,2^n)-3w(2^n,2^n)-4\,y(2^n,2^n)-7^5(2Mer_{5n}+2)=0$

6. $R(2^n,2^n)-98g_{2^n}-98=0$

SOLUTION PATTERN-3

Write (3) as $(v^2+3u^2)=(7)(R^5)(1)$ -- (10)

Write $1=\left(\frac{(1+i\sqrt{3})\left((1-i\sqrt{3})\right)}{4}\right)$ (11)

Using (4), (5), (11) in (10) and by the method of factorization

$$\left(v+i\sqrt{3}\,u\right)=\frac{1}{2}\left(1+i\sqrt{3}\right)\left(2+i\sqrt{3}\right)\left(b+i\sqrt{3}\,a\right)^5$$

Equating the real and imaginary parts,

$u = u(a,b)$

$$= \frac{1}{2}(-5b^4a + 30b^2a^3 - 9a^5 + 3b^5 - 90b^3a^2 + 135ba^4)$$

$v = v(a,b)$

$$= \frac{1}{2}(-45b^4a + 270b^2a^3 - 81a^5 - b^5 + 30b^3a^2 - 45ba^4) \quad (12)$$

Substituting u and v from (12) in (2) and from (4), we get

$$x = x(a,b) = \frac{1}{2}(-50b^4a + 300b^2a^3 - 90a^5 + 2b^5 - 60b^3a^2 + 90ba^4)$$

$$y = y(a,b) = \frac{1}{2}(40b^4a - 240b^2a^3 + 72a^5 + 4b^5 - 120b^3a^2 + 180ba^4)$$

$$z = z(a,b) = \frac{1}{2}(-55b^4a + 330b^2a^3 - 99a^5 + 5b^5 - 150b^3a^2 + 225ba^4)$$

$$w = w(a,b) = \frac{1}{2}(35b^4a - 210b^2a^3 + 63a^5 + 7b^5 - 210b^3a^2 + 315ba^4)$$

$$R = R(a,\ b) = b^2 + 3a^2$$

For the integral solution replace a by 2A and b by 2B,

$x = x(A,B)$

$$= 2^4(-50B^4A + 300B^2A^3 - 90A^5 + 2B^5 - 60B^3A^2 + 90BA^4)$$

$$y = y(A,B) = 2^4(40B^4A - 240B^2A^3 + 72A^5 + 4B^5 - 120B^3A^2 + 180BA^4)$$

$z = z(A,B)$

$$= 2^4(-55B^4A + 330B^2A^3 - 99A^5 + 5B^5 - 150B^3A^2 + 225BA^4)$$

$$w = w(A,B) = 2^4(35B^4A - 210B^2A^3 + 63A^5 + 7B^5 - 210B^3A^2 + 315BA^4)$$

$R = R(A,B) = 4B^2 + 12A^2$, where A, B are arbitrary constants

PROPERTIES:

1. $x(A,A) - y(A,A) - z(A,A) = 0$
2. $w(A,A) = 0$
3. $x(A,A) + y(A,A) + z(A,A) \equiv 0(mod6)$
4. $x(A,A) + y(A,A) \equiv 0(mod2)$
5. $z(A,A) + 4y(A,A) = 0$
6. $x(A,A) - y(A,A) + z(A,A) \equiv 0(mod2)$
7. $x(A,A) + 3y(A,A) = 0$
8. $z(2^n,2^n) - 2^{12}Mer_{5n} - 2^{12} = 0$
9. $y(2^n,2^n) + 2^{10}Mer_{5n} + 2^{10} = 0$

SOLUTION PATTERN-4

Now write $7 = \frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}$ and

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (13)$$

Using (4), (13) in (10) and by the method of factorization

$$(v + i\sqrt{3}\,u)$$

$$= \frac{1}{4}(5 + i\sqrt{3})(1 + i\sqrt{3})(b + i\sqrt{3}\,a)^5$$

Equating the real and imaginary parts,

$$u = u(a,b) = \frac{1}{2}(5b^4a - 30b^2a^3 + 9a^5 + 3b^5 - 90b^3a^2 + 135ba^4)$$

$$v = v(a,b) = \frac{1}{2}(-45b^4a + 270b^2a^3 - 81a^5 + b^5 - 30b^3a^2 + 45ba^4) \quad (14)$$

Substituting u and v from (14) in (2) and from (4), we get

$$x = x(a,b) = \frac{1}{2}(-40b^4a + 240b^2a^3 + 54a^5 + 4b^5 - 120b^3a^2 + 180ba^4)$$

$$y = y(a,b) = \frac{1}{2}(50b^4a - 300b^2a^3 + 90a^5 + 2b^5 - 60b^3a^2 + 90ba^4)$$

$$z = z(a,b) = \frac{1}{2}(-35b^4a + 210b^2a^3 - 63a^5 + 7b^5 - 210b^3a^2 + 315ba^4)$$

$$w = w(a,b) = \frac{1}{2}(55b^4a - 330b^2a^3 + 99a^5 + 5b^5 - 150b^3a^2 + 225ba^4)$$

$$R = R(a,\ b) = b^2 + 3a^2$$

For the integral solution replace a by 2A and b by 2B,

$$x = x(A,B) = 2^4(-40B^4A + 240B^2A^3 + 54A^5 + 4B^5 - 120B^3A^2 + 180BA^4)$$

$$y = y(A,B) = 2^4(50B^4A - 300B^2A^3 + 90A^5 + 2B^5 - 60B^3A^2 + 90BA^4)$$

$$z = z(A,B) = 2^4(-35B^4A + 210B^2A^3 - 63A^5 + 7B^5 - 210B^3A^2 + 315BA^4)$$

$$w = w(A,B) = 2^4(55B^4A - 330B^2A^3 + 99A^5 + 5B^5 - 150B^3A^2 + 225BA^4)$$

$R = R(A,B) = 4B^2 + 12A^2$, where A, B are arbitrary constants

PROPERTIES:

1. $y(A,A) + z(A,A) + w(A,A) = 0$
2. $x(A,A) + z(A,A) \equiv 0 mod(542)$
3. $4y(A,A) + x(A,A) + z(A,A) \equiv 0 mod(6)$
4. $y(A,A) + w(A,A) \equiv 0 mod(2)$
5. $R(A,A)$ *is a perfect square*.
6. $R(2^n, 2^n) - 16CarI_n - 16Mer_{n+1} - 32 = 0$

3. CONCLUSION

In this paper I have presented infinitely many integral solutions of a Sextic Diophantine equation of degree in five unknowns. To conclude one may search for some other pattern of solutions and their corresponding properties.

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AUTO-BIOGRAPHY

Chikkavarapu Gnanendra Rao, (born on 31-1-1983 in Nuzvid, Andhra Pradesh) completed post-graduation in mathematics from P.B.Siddartha College of arts and science in Vijayawada in the year 2005. He worked as assistant professor of mathematics in SSIET NUZVID for ten years. Presently he is working as assistant professor of mathematics in RGUKT IIIT SRIKAKULAM at Prefab Nuzvid Campus. He was qualified in GATE 2013 and APSET-2012.