On Greatest Common Divisor and its Application for a Geometrical Structure

A.K. Mule¹, J.N. Salunke²

¹Department of Mathematics MGM College Ahmedpur, Tq. Ahmedpur Dist. Latur, Maharashtra, India
²Former Director School of Mathematical Sciences SRTMU Nanded at Post Khadgaon, Tq. Dist. Latur, Maharashtra, India

Abstract - In this article some results about GCD are discussed and we derive a formula for smallest number of identical cuboids to construct a square floor/ a cube.

Key Words: GCD, LCM, prime, relatively prime, cuboid.

1. INTRODUCTION

\( \mathbb{N} = \{1,2,3,... \} \), \( \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3,... \} \) are sets of natural numbers and integers respectively. A nonzero integer \( b \) is a divisor (factor) of \( a \in \mathbb{Z} \) if \( a = kb \) for some \( k \in \mathbb{Z} \) and in this case we write \( b|a \) and in this case \( |b|\in \mathbb{Z} \) if \( a\neq 0 \). Divides’‘ is areflective, transitive relation on a set of nonzero integers and it is partial order on \( \mathbb{N} \).

For integers \( a, b \) (not both zero); \( d \in \mathbb{N} \) is called a greatest common divisor (GCD) of \( a, b \) denoted by \( (a,b) \) or \( \text{GCD} \ (a,b) \) if

i) \( d|a, d|b \) and
ii) for any integer \( e \) with \( e|a \) and \( e|b \Rightarrow e|d \)

If \( (a,b) = 1 \) then \( a \) and \( b \) are called relatively prime (coprimes).

For integers \( a, b, c \) (not all zero); \( d \in \mathbb{N} \) is called a GCD of \( a, b, c \) denoted by \( \text{GCD} \ (a,b,c) \) if

i) \( d|a, d|b, d|c \) and
ii) For \( e \in \mathbb{Z} ; e|a, e|b \) and \( e|c \Rightarrow e|d \).

Similarly we define GCD of four or more integers.

For non-zero integers \( a, b; \ell \in \mathbb{N} \) is a least common multiple (LCM) of \( a, b \) and denoted by \( [a,b] \) if

i) \( a|\ell, b|\ell \) and
ii) \( a|m \) and \( b|m \) for \( m \in \mathbb{Z} \Rightarrow \ell|m \)

Similarly we define LCM of three or more nonzero integers.

Note that \( (a,b) = (b,a) = (|a|, |b|), [a,b] = [b,a] = ([|a|,|b|] \).

1.1 Euclid’s Lemma:

For \( a\neq 0 \), \( b, c \in \mathbb{Z}; \) \( bc \) and \( (a,b) = 1 \) then \( a|c \).

1.2 If \( a, b \) are non-zero integers and \( d, k \in \mathbb{Z} \) then

\( (ka, kb) = k(a, b) \) and \( (a,b) = \text{diff} (\frac{a}{a}, \frac{b}{b}, k) = 1 \).

1.3 If \( a, b, c \in \mathbb{Z} \) and \( (a,b) = (a,c) = 1 \) then \( (a,bc) = 1 \).

1.4 Let \( a, b \in \mathbb{N} \) and \( d=(a,b) \). Then

i) There exist \( p, q \in \mathbb{N} \) with \( \frac{a}{p}, \frac{b}{q} \)

\( qd \) and \( (p,q) = 1 \); \n
ii) \( e = pqd = [a,b] \) and

iii) \( \alpha = \beta \)

1.5 [1] For any nonzero integers \( a, b, h, k; (ah, bk) = (a, b) \) \( (h, k) \)

\( \frac{a}{(a,b)} (h,k) \)

\( \frac{b}{(a,b)} (h,k) \)

In particular \( (ah, bk) = (a, k) (b, h) \) if \( (a, b) = (h, k) = 1 \).

Proof

Let \( d = (a, b) = (k, h) \). Then there exist \( p, q, r, s \in \mathbb{Z} \) with \( a = pd, b = qd, h = rf, k = sf \) and \( (p, q) = (r, s) = 1 \).

Hence \( (ah, bk) = \text{prdf}, \text{qsdf} = \text{df} \) \( (pr, qs) \).

Let \( a = (p, s), \beta = (q, r) \). Then \( \exists p_1, s_1, q_1, r_1 \in \mathbb{Z} \) such that

\( p = p_1, a, s = s_1a, q = q_1a, r = r_1a \) and \( (p_1, s_1) = (q_1, r_1) = 1 \).

\( (p, s) \), \( (q, r) \) since \( (p_1, q_1) = (r_1, s_1) \) \( = q \) as \( (p, q) = (r, s) = 1 \) and \( p_1|p, q_1|q, r_1|r, s_1|s \).

Now \((r_1, q_1) = (p_1, s_1) = 1\) gives \((p_1 r_1, q_1) = 1\) and hence \((p_1 r_1, q_1 s_1) = 1\) and \((**)\) follows. Using \((**)\) in \((*)\), we get \((\alpha h, k) = df a\beta = (\alpha, b)(h, k)(p, s)(q, r) \)

\[
= \frac{\alpha}{(a, b)} \frac{k}{(h, k)} = b(h, k)\]

as \(p = \frac{a}{d} (a, b), s = \frac{k}{f} (h, k)\) etc.

For more details and proofs of above results, one may refer [2], [3] or any standard book on Elementary Number Theory.

2. Application 1

Now we derive a formula for requiring least number of identical rectangular tiles which need to pave a square floor without breaking any tile.

2.1 Proposition:

If \(a, b \in \mathbb{N}\) and there are rectangular tiles of size \(a \times b\) (sq. units) then to form a square of smallest size by fitting these tiles requires \(\frac{ab}{d^2}\) tiles, where \(d = (a, b)\). Moreover for any \(k \in \mathbb{N}\), fitting \(\frac{k^2 ab}{d^2}\) tiles we form a square.

Proof:

Let \(d = (a, b)\) where \(a, b \in \mathbb{N}\). Then \(\frac{a}{d}, \frac{b}{d} \in \mathbb{N}\). Now form a row of \(\frac{b}{d}\) tiles, where each tile is of length \(\frac{a}{d}\) and width \(\frac{b}{d}\).

This row forms a rectangle of size \(\frac{ab}{d^2} \times \frac{b}{d}\).

Consider \(\frac{a}{d}\) such rows. Thus we have \(\frac{b}{d} \times \frac{a}{d}\) tiles forming \(\frac{b}{d}\) rows and \(\frac{b}{d}\) columns, forming a rectangular floor and each side of this rectangle is \(\frac{ab}{d^2}\), i.e. the rectangle is a square of side \(\frac{ab}{d}\).

The number of tiles in this square floor is \(\frac{ab}{d} \times \frac{a}{d}\).

[Area of the square = \(\frac{(ab)^2}{d^2} = \frac{ab}{d} \times \frac{ab}{d}\) (Area of a tile).(number of tiles)].

Let \(X\) be any square formed by tiles. Let \(t, s\) be number of tiles in row and column respectively in the square \(X\), i.e. \(X\) is formed by these tiles.

Now four sides of \(X\) are equal gives \(at = bs\)

\[
\frac{a}{d} t = \frac{b}{d} s \quad \text{and hence} \quad \frac{a}{d} \frac{b t}{d} = \frac{b}{d} \frac{a t}{d}
\]

\[
\frac{a}{d} s = \frac{b}{d} t \quad \text{since} \quad \frac{a}{d} = 1 \quad \text{(by 1.1)}
\]

\[
\frac{a}{d} \leq s \quad \text{and} \quad \frac{b}{d} \leq t, \quad \text{i.e.} \quad \frac{ab}{d^2} \leq st = \text{Number of tiles in} \ X
\]

This proves that the smallest square is formed with \(\frac{ab}{d^2}\) tiles.

Note that such 4 squares, 9 squares, form squares.

Illustration 1: The smallest square floor which can be completely paved with tiles of size 8x6, without breaking any tile needs \(\frac{8x6}{(8,6)^2} = \frac{48}{4} = 12\) tiles.

3. Relation between LCM and GCD of three numbers.

For any non-zero integers \(a, b, c\) and \(d \in \mathbb{N}\), we have

\[
(\alpha, b, c) = (b, c, \alpha) = (c, \alpha, b) = (\alpha, c, b) = (|\alpha|, |b|, |c|).
\]

\[
[a, b, c] = [b, c, a] = [c, a, b] = [a, c, b] = [|\alpha|, |b|, |c|].
\]
(\(ka, kb, kc\)) = k(\(a, b, c\)), [\(ka, kb, kc\)] = k[\(a, b, c\)] and

\((a, b, c) = d \iff \left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right) = 1\)

### 3.1 Theorem:
Let \(a, b, c \in \mathbb{N}\) and \(d = (a, b, c)\), \(d_1 = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d})\), \(d_2 = (\frac{b}{d}, \frac{c}{d})\), \(d_3 = (\frac{a}{d}, \frac{b}{d})\). Then

**i)** There exist \(a, b, c \in \mathbb{N}\) such that \(a = dd_1a_1\), 
\(b = dd_1db_1, c = dd_2dc_1;\)

**ii)** \((d_1, d_2) = (d_2, d_3) = (d_1, d_3) = 1\)

**iii)** \((a_1, d_1) = (b_1, d_1) = (c_1, d_1) = 1\)

**iv)** \((a_1, b_1) = (b_1, c_1) = (a_1, c_1) = 1\)

**v)** \([a, b, c] = dd_1dd_2dd_3a_1b_1c_1\) is the LCM of \(a, b, c\).

**vi)** \(abc = [a, b, c](a, b, c)^2\)

**Proof:** Let \(a, b, c \in \mathbb{N}\), \(d = (a, b, c)\), \(d_1 = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d})\), \(d_2 = (\frac{b}{d}, \frac{c}{d})\), \(d_3 = (\frac{a}{d}, \frac{b}{d})\).

- **ii)** Let \(h = (d_1, d_2)\). Then \(h|d_1, h|d_2\) \((d_1, d_2) = (\frac{a}{d}, \frac{b}{d})\) it gives

  \(h|\left(\frac{a}{d}\right), h|\left(\frac{b}{d}\right)\) and hence \(h\) divides \((\frac{a}{d}, \frac{b}{d}) = 1\),

  i.e. \(h = 1 = (d_1, d_2)\). Similarly \((d_2, d_3) = (d_1, d_3) = 1\).

- **iii)** Now \(d_1|\left(\frac{a}{d}\right), d_2|\left(\frac{b}{d}\right)\) and \((d_2, d_3) = 1 \Rightarrow d_3|\left(\frac{a}{d}\right)\) by 1.3.

  Hence there exists \(a_1 \in \mathbb{N}\) such that \(\frac{a}{d} = dd_1a_1\)

  i.e. \(a = dd_1dd_3a_1\). Similarly there exist \(b_1, c_1 \in \mathbb{N}\)

- **iii)** Let \(g = (a_1, d_2)\). Then \(g|\left(\frac{a}{d}\right), g|\left(\frac{b}{d}\right), g|\left(\frac{c}{d}\right)\) as \(a_1|\left(\frac{a}{d}\right)\) and \(d_2 = (\frac{b}{d}, \frac{c}{d})\).

  \[\Rightarrow g\) divides \((\frac{a}{d}, \frac{b}{d}) = 1, i.e. (a_1, d_2) = g = 1.\]

  Similarly \((b_1, d_1) = (c_1, d_1) = 1\).

- **iv)** Let \(f = (a_1, b_1)\). Then \(f|\left(\frac{a}{d_1}\right), f|\left(\frac{b}{d_1}\right)\) by (i)

  \[\Rightarrow f\) divides \((\frac{a}{d_1}, \frac{b}{d_1}) = 1, since d_1 = (\frac{a}{d}, \frac{b}{d});\]

  i.e. \((a_1, b_1) = f = 1. Similarly \((b_1, c_1) = (a_1, c_1) = 1.\)

- **v)** \(\ell = dd_1dd_2dd_3a_1b_1c_1\) and \(\ell = dd_2b_1c_1 = dd_3a_1c_1 = cd_1a_1b_1\) by (i)

  \[\Rightarrow a|\ell, b|\ell, c|\ell.\]

  Let \(m \in \mathbb{Z}\) be such that \(a|m, b|m, c|m\). Then by (i),

  \[d_1d_3a_1|\frac{m}{d}, d_1d_2b_1|\frac{m}{d}, d_2d_3c_1|\frac{m}{d};\]

  Now, \((d_1, c_1) = 1 = (d_1, d_2d_3)\) by (ii) and (iii) \(\Rightarrow (d_1, d_2d_3c_1) = 1.\)

  \[\Rightarrow d_1d_2d_3c_1|\frac{m}{d}\) as \(d_1|\frac{m}{d}\) and \(d_2d_3c_1|\frac{m}{d}\)

  Similarly \(d_1d_2d_3a_1|\frac{m}{d}, d_1d_2d_3b_1|\frac{m}{d}^{2}\)

  Thus \(a_1|\frac{m}{d}, b_1|\frac{m}{d}, c_1|\frac{m}{d}\)

  As \((a_1, c_1) = (b_1, c_1) = 1 \Rightarrow (a_1b_1, c_1) = 1, so\)

  above we have \(a_1b_1c_1|\frac{m}{d}\) i.e. \(dd_1dd_2dd_3a_1b_1c_1|m.\)

  i.e. \(\ell = dd_1dd_2dd_3a_1b_1c_1\) divides \(m.\)

  Hence \(\ell = dd_1dd_2dd_3a_1b_1c_1 = [a, b, c] \) is the LCM of \(a, b, c\).

- **iii)** \(abc = d^2(d_1d_2d_3)^2a_1b_1c_1\)

  \[= [a, b, c](a_1b_1c_1)^2(a, b, c)(a, c).\]

### 3.2 Corollary: If \(a, b, c \in \mathbb{N}\) and \(d = (a, b, c)\) such that \(a = dd_1a_1, b = dd_2b_1, c = dd_3c_1,\)

\(a, b, c \in \mathbb{N}\)

\(b = db_1, c = dc_1\) and \((a_0, b_0) = (b_0, c_1) = (a_1, c_1) = 1\)

\(abc = [a, b, c](a, b, c)^2.\)

Proof follows from theorem 3.1, since here \(d_1 = d_2 = d_3 = 1.\)

Cuboid of size \(abcx\) is a solid rectangular parallelepiped of length \(a, width b and height c. (units).\)
3.3 Proposition:

If $a$, $b$, $c \in \mathbb{N}$ and there are cuboids of same size $a \times b \times c$ (cubic units), then using $\frac{(abc)^2}{(d^2 d_1 d_2 d_3)^3}$ cuboids we can form a solid cube. Moreover for any $k \in \mathbb{N}$, using $\frac{k^3 (abc)^2}{(d^2 d_1 d_2 d_3)^3}$ cuboids, we can form a solid cube, where $d = (a, b, c)$.

$$d_1 = \left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right), \quad d_2 = \left(\frac{b}{d}, \frac{c}{d}, \frac{a}{d}\right), \quad d_3 = \left(\frac{c}{d}, \frac{a}{d}, \frac{b}{d}\right).$$

Proof: Let $a$, $b$, $c \in \mathbb{N}$; $A = d_1 d_2 d_3 a_1$, $b = d_1 d_2 d_3 b_1$, $c = d_1 d_2 d_3 c_1$

Where $d = (a, b, c)$, $d_1 = \left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right)$, $d_2 = \left(\frac{b}{d}, \frac{c}{d}, \frac{a}{d}\right)$, $d_3 = \left(\frac{c}{d}, \frac{a}{d}, \frac{b}{d}\right)$ and $a_i, b_i, c_i \in \mathbb{N}$.

Now consider the row.

Consider cuboids as tiles of sizes $\alpha \times \beta \times \gamma$ with thickness (height) $c$.

Now we form a square floor with $\frac{bc}{d^2 d_1 d_2 d_3}$ columns and $\frac{ac}{d^2 d_1 d_2 d_3}$ rows of tiles.

[Note that $\frac{bc}{d^2 d_1 d_2 d_3}, \frac{ac}{d^2 d_1 d_2 d_3}, \frac{ab}{d^2 d_1 d_2 d_3} \in \mathbb{N}$.]

Thus we have a cuboid with square base of side $\frac{abc}{d^2 d_1 d_2 d_3}$ and height $c$.

Now consider such cuboids (with square base) and installing them we get a solid cube of edge $\frac{abc}{d^2 d_1 d_2 d_3}$.

For this forming the solid cube we require

$$\frac{bc}{d^2 d_1 d_2 d_3} \times \frac{ac}{d^2 d_1 d_2 d_3} \times \frac{ab}{d^2 d_1 d_2 d_3} = \frac{(abc)^2}{(d^2 d_1 d_2 d_3)^3}$$

cuboids. □

Remark:

In proposition 3.2, if $(d_3, c_1) = (d_3, b_1) = (d_1, b_1) = 1$.

Then number of cuboids required to form smallest cube is $\frac{(abc)^2}{(d^2 d_1 d_2 d_3)^3}$.

Let $X$ be a cube formed from cuboids each of size $\alpha \times \beta \times \gamma$. Let $r, s, t$ be a number of cuboids along the sides of the cube $X$. As each edge of $X$ is same, we have

$$ar = bs = ct,$$

i.e. $d = (a, b, c)$.

$$d_1 = \left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right), \quad d_2 = \left(\frac{b}{d}, \frac{c}{d}, \frac{a}{d}\right), \quad d_3 = \left(\frac{c}{d}, \frac{a}{d}, \frac{b}{d}\right).$$

Proof: Let $a, b, c \in \mathbb{N}$; $A = d_1 d_2 d_3 a_1$, $b = d_1 d_2 d_3 b_1$, $c = d_1 d_2 d_3 c_1$.

Now $d_1 d_2 d_3 a_1 r$ with $(d_2, d_3) = (d_2, a_1) = 1$, so $(d_2, d_3 a_1) = 1$,

$(b_1, a_1) = 1 = (b_1, d_3) = 1$, so $(b_1, d_3 a_1) = 1$ (See theorem 3.1)

and hence $(d_2 b_1, d_3 a_1) = 1$, which gives by 1.1, $d_2 b_1 | r$.

Also we have $c_1 | d_1 a_1 r$ and $(c_1, d_1) = (c_1, a_1) = 1$,

i.e. $(c_1, d_1 a_1) = 1$, so again by 1.1, $c_1 | r$.

$d_2 b_1 | r, c_1 | r$ and $(b_1, c_1) = 1$, $(d_2, c_1) = 1$ i.e. $(d_2 b_1, c_1) = 1$ gives $d_2 b_1 c_1 | r$ (by 1.1). Now $\frac{bc}{d^2 d_1 d_2 d_3} = d_2 b_1 c_1$ divides $r$.

Thus $\frac{bc}{d^2 d_1 d_2 d_3} \leq r$.

Using $(d_3, c_1) = (d_1, b_1) = 1$, we get

$$\frac{ac}{d^2 d_1 d_2 d_3} = d_2 a_1 c_1 \leq s, \quad \frac{ab}{d^2 d_1 d_2 d_3} = d_1 a_1 b_1 \leq t.$$

Hence $\frac{abc^2}{(d^2 d_1 d_2 d_3)^3} \leq rst = \text{Number of cuboids in X.}$

This proves that the smallest cube is formed with $\frac{(abc)^2}{(d^2 d_1 d_2 d_3)^3}$ cuboids.

If $d_1 = d_2 = d_3 = 1$ then $\frac{ab^2 c^2}{(d^2 d_1 d_2 d_3)^3} = \frac{abc^2}{(d^2 d_1 d_2 d_3)^3}$

3.4 Corollary [Application 2]

Let $a, b, c \in \mathbb{N}$; $d = (a, b, c)$, $d_1 = \left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right) = d_1 d_2 d_3 a_1$, $b = d_1 d_2 d_3 b_1$, $c = d_1 d_2 d_3 c_1$. If one of the following holds:

$(d_1, a_1) = 1$, $(d_1, b_1) = 1$, $(d_2, b_1) = 1$, $(d_2, c_1) = 1$, $(d_3, a_1) = 1$, $(d_3, c_1) = 1$.
then using \( \frac{(abc)^2}{(d^2d_1d_2d_3)^3} \) cuboids, each of size \( a \times b \times c \) (cubic units), we can form the smallest cube without breaking any cuboid.

Hint: In the above remark, for \((d_1, c_1) = 1\), we have \( \frac{bc}{d^2d_1d_2d_3} \leq r \), i.e., \( \Rightarrow \frac{abc}{d^2d_1d_2d_3} \leq \) Volume of the cube \( X \)

From this corollary 3.4 follows.

Illustration 2: Determination of the number of cuboids, each of size 100x210x375 (cubic units) to form the smallest solid cube without breaking any cube.

For \( a = 100 = 5 \times 2 \times 5 \times 2, b = 210 = 5 \times 2 \times 3 \times 7, c = 375 = 5 \times 3 \times 5 \times 5 \) \( \in \mathbb{N} \).

We have \( d = (a, b, c) = 5, d_1 = (\frac{a}{d}, \frac{b}{d}) = 2, d_2 = (\frac{b}{d}, \frac{c}{d}) = 3, d_3 = (\frac{c}{d}, \frac{a}{d}) = 5, \alpha_1 = 2, b_1 = 7, c_1 = 5, \) with \( d_1, d_2, d_3 \) are pairwise relatively prime and \( \alpha_1, b_1, c_1 \) pairwise relatively prime with \((\alpha_1, d_3) = (b_1, d_3) = (c_1, d_1) = 1 \) (verifies theorem 3.1).

Note that \((d_1, a_1) = 2, (d_1, b_1) = 1 = (d_2, b_1) = (d_2, c_1) = (d_3, \alpha_1) = 1 \) and \( (d_3, c_1) = 5 \).

So by corollary 3.4, to form smallest cube from cuboids, each of size \( a \times b \times c \) (cubic units) without breaking any cuboid requires number of cuboids

\[
\frac{abc}{(d^2d_1d_2d_3)^3} = \frac{100 \times 210 \times 375}{(5^2 \times 2 \times 3 \times 5)^3} = \frac{(7875000)^2}{(750)^2} = 147000.
\]

3.5 Corollary. Let \( a, b, c \in \mathbb{N}, d = (a, b, c) \) and \( a = d\alpha_1, b = d\beta_1, c = d\gamma_1 \) with

\[(\alpha_1, b_1) = (b_1, c_1) = (c_1, d_1) = 1.\]

Using \( \frac{abc^2}{(d)^4} \) cuboids, each of size \( a \times b \times c \), we can form smallest solid cube. For any \( k \in \mathbb{N} \), using \( k^3 (a \beta_1 c_1)^3 \)

such cuboids we form a solid cube.

Illustration 3: To form smallest cube from cuboids each of size 10x6x4, we require

\[
\frac{(10 \times 6 \times 4)^2}{(10 \times 6 \times 4)^4} = \frac{(10 \times 6 \times 4)^2}{2^6} = (5 \times 3 	imes 2)^2 = 900.
\]