EFFICIENT BONDAGE NUMBER OF A JUMP GRAPH

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ABSTRACT: A set $S$ of vertices in jump graph $J(G)$ is an efficient domination set, if every vertex in $V-S$ is adjacent exactly one vertex in $S$. The efficient domination number $\gamma_e(J(G))$ of $J(G)$ is minimum number of vertices is an efficient dominating set of $J(G)$. In general $\gamma_e(J(G))$ can be made to increase by removal of edges from $J(G)$. Our main objective is to Study this phenomenon. Let $E$ be set of edges of $J(G)$ such that $\gamma_e(J(G)-E) > \gamma_e(J(G))$. Then we define the efficient bondage number $b_e(J(G))$ of $J(G)$ to be the minimum number of edges in $E$. In this communication an upper bound for $b_e(J(G))$ has been established and its exact values for some classes of graph have been found. In addition Nordhaus-Gaddum type results are established.

Key words: dominating set, bondage number.

Mathematical classification: b05C56.

1. INTRODUCTION: Dominating sets were studied by Berge.C[1] and ore[2] Domination alteration sets in graphs were studied by Bauer et.al[3] . A similar concept named as the bondage number of a graph was studied by Fink et.al,[4]. The efficient domination number was introduced by Cockayne et.al., [5]. In this communication we study stability of $\gamma_e(J(G))$ by defining the efficient bondage number $b_e(J(G))$ of a jump graph $J(G)$. The graphs considered in this communication are finite undirected, without loops, multiple edges and isolated vertices. Any undefined terms here may be found in Harary [6]. A set $X$ of vertices is a dominating set of $J(G)$ if every vertex in $X$ is adjacent at least one vertex in $X$. The domination number $\gamma(J(G))$ of $J(G)$ is the minimum number of vertices in a dominating set of $J(G)$. Let $E$ be a set of edges such that $\gamma(J(G)-E) > \gamma(J(G))$. Then the bondage number $b(J(G))$ of $J(G)$ is the minimum number of edges in $E$. A set $S$ of vertices in $J(G)$ is an efficient dominating set if every vertex $u$ in $V-S$ is adjacent exactly one vertex in $S$. The efficient domination number $\gamma_e(J(G))$ of $J(G)$ is the minimum number of vertices is an efficient dominating set of $J(G)$.

Let $E$ be a set of edges such that $\gamma_e(J(G)-E) > \gamma_e(J(G))$. Then we can define the efficient bondage number $b_e(J(G))$ is the minimum number of edges in $E$. Here we note that if $\gamma_e(J(G)) = p$ then $b_e(J(G))$ does not exists.

2. Results:

The following results is straight forward hence we omit the proof.

**Theorem1:** for any graph $J(G)$ with $p$ vertices

$\gamma_e(J(G)) = 1$ if and only if $\Delta(J(G))=p-1$.

**Theorem 2:** For any path $P_p$ with $p=2$ vertices

$\gamma_e(J(P_p)) = \lceil p/3 \rceil$

**Theorem 3:** For any cycle $C_p$ with $p \geq 3$ vertices

$\gamma_e(C_p) = \lceil \frac{p}{3} \rceil$ if $p \equiv 0,1 \pmod{3}$

$= \lceil \frac{p}{3} \rceil = 1$ if $p \equiv 2 \pmod{3}$

Hence $\lceil x \rceil$ denotes the least integer greater than or equal to $x$. 

3. Main Results

**Theorem 3.1:** Let \( J(G) \) be a graph \( \Delta(J(G)) = p - 1 \) Then \( b_e(J(G)) = \lceil \frac{n}{2} \rceil \gamma \) where \( n \) is the number of vertices of degree \( p - 1 \).

**Proof:** Let \( u_1, u_2, u_3, ..., u_n \) be the \( n \) vertices of degree \( p - 1 \) then clearly removal of fewer than \( \lceil \frac{n}{2} \rceil \) edges results into a graph \( J(G') \) having maximum degree \( \Delta(G') = p - 1 \).

Hence \( b_e(J(G)) \geq \lceil \frac{n}{2} \rceil \gamma \)

Now we consider the following cases.

**Case (i):** If \( n \) is even then the removal of \( \frac{n}{2} \) independent edges \( u_1u_2, u_3u_4, ..., u_{n-1}u_n \) results into a graph \( J(H) \) having \( \Delta(J(G)) = p - 2 \) Hence \( b_e(J(G)) = \frac{p}{2} \).

**Case (ii):** If \( n \) is odd then the removal of \( \frac{n-1}{2} \) independent edges \( u_1u_2, u_3u_4, ..., u_{n-2}u_{n-1} \) yields a graph \( J(H') \) containing exactly one vertex \( u_n \) of degree \( p - 1 \). Thus by removing an edge incident with \( u_n \) we obtain a graph \( J(H'') \) with \( \Delta(J(H'')) = p - 2 \) \( \gamma_e(J(G')) \geq 2 \).

Hence from case (i) and (ii) it follows that

\[
\begin{align*}
\gamma_e(J(G)) &= \frac{n}{2} \text{ if } p \text{ is even} \\
&= \frac{n-1}{2} + 1 \text{ if } p \text{ is odd} \\
\gamma_e(J(G)) &= \lceil \frac{n}{2} \rceil \gamma \\
\end{align*}
\]

Hence the proof.

The following result directly from Theorem 3.1

**Proposition 3.2:** For any complete graph \( K_p \) with \( p \geq 2 \) vertices \( b_e(J(K_p)) = \gamma_e(J(K_p)) = \frac{p}{2} \gamma \)

**Proof:** By theorem 3.1 \( b_e(K_p) = \frac{n}{2} \) since \( n=p \)

**Proposition 3.3:** For any wheel \( W_p \) with \( p \geq 5 \) vertices \( b_e(J(W_p)) = 1 \)

**Proof:** Since \( W_p \) contains exactly one vertex of degree \( p - 1 \) Hence

\[
\gamma_e(J(W_p)) = \frac{1}{2} \gamma = 1
\]

**Theorem 3.4:** Let \( K_{m,n} \) be a complete bipartite graph other then \( C_3 \) with \( 1 \leq m \leq n \) then

\[
\gamma_e(J(K_{m,n})) = m
\]

**Proof:** Let \( v = v_1 \cup v_2 \) be the vertex sets of \( K_{m,n} \) where \( |v_1| = m \) and \( |v_2| = n \) let \( v \in v_2 \) then by removing all edges incident with \( v \) we obtain a graph \( J(G') \) containing two components

\[ K_1 \text{ and } K_{m,n-1} \]

Hence \( \gamma_e(J(G')) = \gamma_e(J(K_1)) + \gamma_e(J(K_{m,n-1})) \)

\[ = 1 + \gamma_e(J(K_{m,n})) \geq \gamma_e(J(K_{m,n})) \]
Thus \( \text{be}(J(K_{m,n})) = \deg v = |v_1| = m \)

**Proposition 3.5**: For any cycle \( C_p \) with \( p \geq 3 \) vertices

\[
\text{be}(J(C_p)) = 2 \text{ if } p \equiv 0 \pmod{3} \\
= 3 \text{ if } p \equiv 1 \pmod{3} \\
= 4 \text{ if } p \equiv 2 \pmod{3}
\]

**Proof**: Let \( C_p \) be a cycle with \( p \geq 3 \) vertices. Then we consider the following cases,

**Case 1**: If \( p \equiv 0 \pmod{3} \) let \( J(H) \) be a graph OBTAINED BY REMOVING TWO ADJACENT EDGES FROM \( C_p \). Then clearly \( J(H) \) consists of an isolated vertex and a path of order \( p - 1 \).

Thus \( \gamma_e(J(H)) = 1 + \gamma_e(J(P_{p-1})) = 1 + \sum \frac{p-1}{3} \)

Since \( p \equiv 0 \pmod{3} \) \( \sum \frac{p-1}{3} = \sum \frac{p}{3} \)

Therefore \( \gamma_e(J(H)) = 1 + \sum \frac{p}{3} \)

\[
= 1 + \gamma_e(J(C_p)) > \gamma_e(J(C_p))
\]

Hence \( \text{be}(J(C_p)) = 2 \)

**Case 2**: if \( p \equiv 1 \pmod{3} \) then the removal of three consecutive edges from \( J(C_p) \) results in a graph \( J(H) \) consisting of two isolated vertices and a path of order \( p - 2 \). Hence,

\[
\gamma_e(J(H)) = 2 + \gamma_e(J(P_{p-2}))
\]

\[
= 2 + \sum \frac{p-2}{3} \)

\[
= 1 + \sum \frac{p}{3} \)

\[
= 1 + \gamma_e(J(C_p)) > \gamma_e(J(C_p))
\]

Thus \( \text{be}(J(C_p)) = 3 \).

**Case 3**: if \( p \equiv 2 \pmod{3} \) then by removing four consecutive edges from \( C_p \) we obtain a graph \( J(H) \) containing three isolated vertices and a path of order \( p - 3 \) then

\[
\gamma_e(J(H)) = 3 + \gamma_e(J(P_{p-3})) = 3 + \sum \frac{p-3}{3} \)

\[
= 2 + \sum \frac{p}{3} > \gamma_e(J(C_p))
\]

Hence \( \text{be}(J(C_p)) = 4 \)

Hence the proof.

**Proposition 3.6**: For any path \( P_p \) with \( p \geq 2 \) vertices then

\[
\text{be}(J(P_p)) = 2 \text{ if } p \equiv 1 \pmod{3} \\
= 1 \text{ otherwise.}
\]
**Proof**: Let $P_p$ be a path with $p > 2$ then we consider the following cases,

**Case 1**: if $p \equiv 1 \pmod{3}$ then the removal of two end edges results a graph $J(G')$ containing two isolated vertices and path of order $p - 2$. Hence,

$$
\gamma_e(J(G')) = 2 + \gamma_e((P_{p-2}))
= 2 + \left\lceil \frac{p-2}{3} \right\rceil
= \left\lceil \frac{p}{3} \right\rceil + 1 > \left\lceil \frac{p}{3} \right\rceil = \gamma_e(J(P_p))
$$

Thus $b_e(J(P_p)) = 2$

**Case 2**: If $p \not\equiv 1 \pmod{3}$ then the removed of an edge from $J(P_p)$ results a graph $J(H)$ containing an isolated vertex and a path of order $p - 1$.

$$
\gamma_e(J(H)) = 1 + \gamma_e((P_{p-1}))
= 1 + \left\lceil \frac{p-1}{3} \right\rceil
= \left\lceil \frac{p}{3} \right\rceil = \gamma_e(J(P_p))
$$

Hence $b_e(J(P_p)) = 1$

**Theorem 3.7**: For any connected graph $J(G)$ with $p \geq 2$ vertices $b_e(J(G)) \leq p - 1$.

Further the bound is attained if $G = C_p$ with $3 \leq p \leq 5$

**Proof**: On the contrary suppose $b_e(J(G)) \geq p$ let $E_u$ denote the set of edges incident with a vertex $u$. Then clearly

$$
\gamma_e(J(G - E_u)) \geq \gamma_e(J(G))
$$

Which is a contradiction and $|E_u| \leq p - 1$

Hence $b_e(J(G)) \leq p - 1$

Further for $J(G) = J(C_p)$ with $3 \leq p \leq 5$ it is easy to see that

$$
b_e(J(G)) = p - 1
$$

**Theorem 3.8**: Let $u$ and $v$ be distinct adjacent vertices in a non trivial graph $J(G)$ then

$$
B_e(J(G)) = \min \{ \deg u + \deg v \}
$$

**Proof**: Let $u$ and $v$ be two distinct adjacent vertices of $J(G)$ such that $\deg u + \deg v$ is minimum. Suppose $b_e(J(G)) = \deg u + \deg v$. Let $E_u$ denote the set of edges that are incident with $u$ and $v$. Then clearly $|E| = \deg u + \deg v - 1$ and hence $\gamma_e(J(G - E)) = \gamma_e(J(G))$. Since $u$ and $v$ are isolated vertices in $J(G - E)$, $\gamma_e(J(G - u - v)) = \gamma_e(J(G)) - 2$. Thus for any minimum efficient dominating set $S$ of $J(G)$ such that $|S| + u + v$ is minimum. Hence $b_e(J(G)) \leq \min \{ \deg u + \deg v \}$.

**Corollary 3.8.1**: For any nontrivial graph $J(G)$

$$
b_e(J(G)) \leq \delta(J(G)) + \Delta(J(G))
$$

**Proof**: This follows from the Theorem 3.8.
Now we obtain a Nordhaus-Gaddum type result.

**Theorem 3.9** For any graph J(G)

(i) \( b_e( J(G) ) + b_e( J(\overline{G}) ) \leq 2 (p - 1) \) and

(ii) \( b_e( J(G) ) \cdot b_e( J(\overline{G}) ) \leq 2p (\delta(J(G)) + \Delta(J(G))) \)

**Proof:** By corollary 3.8.1 we have

\[
b_e( J(G) ) \leq \delta(J(G)) + \Delta(J(G))
\]

and

\[
b_e( J(\overline{G}) ) \leq \delta(J(\overline{G})) + \Delta(J(\overline{G}))
\]

Hence

\[
b_e( J(G) ) + b_e( J(\overline{G}) ) = \delta(J(G)) + \Delta(J(G)) + \delta(J(\overline{G})) + \Delta(J(\overline{G}))
\]

\[
= \delta(J(G)) + \Delta(J(G)) + \delta(J(\overline{G})) + (p-1) \cdot \Delta(J(G)) + p - 1 - \delta(J(G))
\]

\[
= 2(p - 1)
\]

Also

\[
b_e( J(G) ) \cdot b_e( J(\overline{G}) ) = \delta(J(G)) + \Delta(J(G)) \cdot \delta(J(\overline{G})) + \Delta(J(\overline{G}))
\]

\[
= \delta(J(G)) + \Delta(J(G)) \cdot (p - 1 - \Delta(J(G)) + p - 1 - \delta(J(G))
\]

\[
= \delta(J(G)) + \Delta(J(G)) \cdot (2(p-1) - \delta(J(G)) - \delta(J(G)))
\]

\[
= 2p (\delta(J(G)) + \Delta(J(G)))
\]

Thus

\[
b_e( J(G) ) + b_e( J(\overline{G}) ) \leq 2 (p - 1) \) and

\[
b_e( J(G) ) \cdot b_e( J(\overline{G}) ) \leq 2p (\delta(J(G)) + \Delta(J(G))).
\]

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