Approach to Fuzzy Controller Design of an inverted pendulum controlled by a DC motor

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Abstract - *This paper shows the fuzzy controller design of an inverted pendulum controlled by a DC motor. The given nonlinear system is modelled by T-S fuzzy model using the concept of sector nonlinearity and parameter uncertainty is added in the system then a fuzzy controller is design using pole placement approach and the stability analysis of fuzzy control system is done. The performance of the given is check and it performed well in the presence of parameter uncertainty.*

Key Words: T-S Fuzzy model, Sector nonlinearity

1. INTRODUCTION

Nonlinear control is a difficult task due to their complex nature as many physical system are nonlinear in nature. The nonlinearities persists in the system as an inherent nonlinearity which represent the imperfections of the physical devices and as an intentional which are deliberately inserted into the system in order to improve performance. So therefore for the analysis and design of complex nonlinear system traditionally we have some classical control technique like transfer function only for linear system, Taylor's series method of model linearization, describing function analysis and many more techniques [1]. The model linearization by using these techniques cannot give an accurate and appropriate results especially when the system's parameter are changing with respect to time, for example the plant, the sensor and the actuator in the control system. The parameter of the complex system is always changing due to different operating points in the nonlinear system hence to deal with this kind of nonlinearities and uncertainties, the traditional control techniques becomes fails. In literature some control techniques like robust control, adaptive control, variable structure control and many more are available to use [2].

In the recent years, many researchers are involved in controlling such uncertain nonlinear system by using fuzzy logic. The fuzzy control system have been proven to be the superior and most cost effective when compare with conventional control especially in controlling complex nonlinear system with parameter uncertainties [3], [4]. The fuzzy system are developed and applied in various control system applications [5]. Fuzzy control system is well recognized and accepted as an universal approximation of complex uncertain nonlinear system in the sense of model linearization. [6], [7].

In paper we derived the fuzzy based model along with fuzzy controller of an inverted pendulum controlled by a DC motor using T-S fuzzy model by using the concept given by sector nonlinearity which convert original uncertain nonlinear system into linear sub-system with uncertainty. Then the linear sub-system is analysed and its performance is checked with system without uncertainty. The paper is organised as: In section 2 we comes with the problem formulation of an inverted pendulum controlled by a DC motor by using T-S fuzzy model with uncertainty and design of fuzzy controller by pole placement method, In section 3 the stability of an approximate linear sub-system using theorem derived by H. K. Lam, F. H. F. Leung, and P. K. S. Tam is given. In section 4 we show the results of uncertain fuzzy control system and finally the conclusion is drawn.

2. PROBLEM FORMULATION

Consider the inverted pendulum controlled by a DC motor as given below which similar to the example is used in [8].

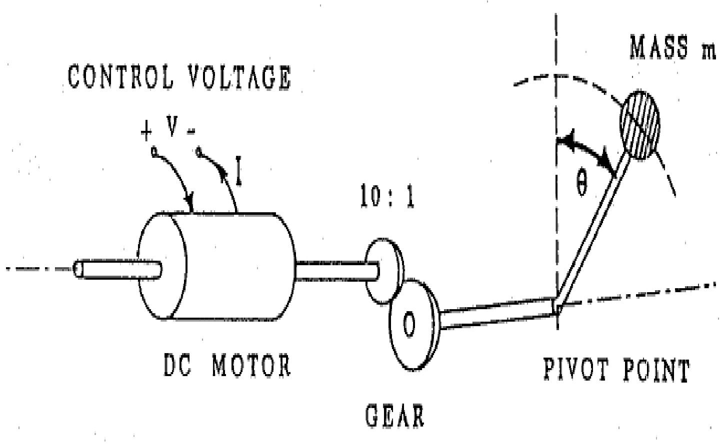

Fig. 1 An Inverted Pendulum controlled by a DC motor

Let $x_1(t)$,$x_2(t)$ and $x_2(t)$ be the state variables which is define as:

$x_1(t) = \theta$ (*angular displacement*); $x_2(t) = \dot{\theta} = \omega$ (angular velocity); $x_3(t) = I$ (current) & $u(t)$ is control input.

Its state equation is given by

$$\dot{x}_1(t) = x_2(t) \tag{1}$$

$$\dot{x}_2(t) = (k_1 + \Delta k_1)\sin x_1(t) + (k_2 + \Delta k_2)x_3(t) \quad (2)$$

$$\dot{x}_3(t) = (k_3 + \Delta k_3)x_2(t) + (k_4 + \Delta k_4)x_3(t) + (k_5 + \Delta k_5)u(t) \quad (3)$$

Where
$k_1 = 9.8;\ k_2 = 1;\ k_3 = -10;\ k_4 = -10\ \&\ k_5 = 10$
and $\Delta k_1, \Delta k_2, \Delta k_3, \Delta k_4\ \&\ \Delta k_5$ are the parameter uncertainties.

2.1 Uncertain T-S Fuzzy model

The nonlinear term in (2) is $\sin x_1(t)$, assume $x_1(t) \epsilon [-a \quad a]$; where $a = \frac{\pi}{2}$. Let $z(t)$ be premise variable define as $z(t) = \sin x_1(t)$, using the method of sector nonlinearity [10] we find its sector as shown as:

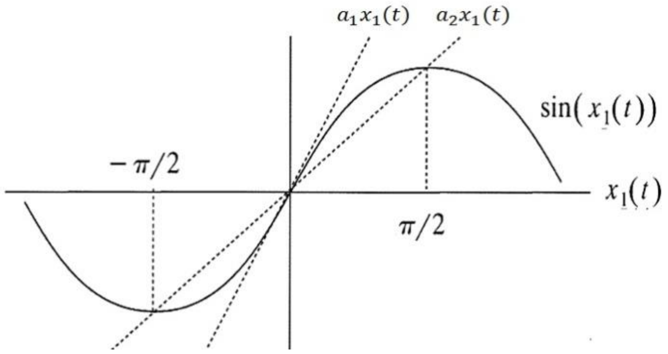

Fig. 2 sin $x_1(t)$ and its sector

From the Fig. 2, we have

$a_1 x_1(t) \leq z(t) \leq a_2 x_1(t)$; where $a_1 = 1\ \&\ a_2 = \frac{2}{\pi}$ are the slopes respectively.

Also; $a_1 x_1(t) \leq \sin x_1(t) \leq a_2 x_1(t)$

Since, $\sin x_1(t) = W_{M_1}(z(t)) * a_1 x_1(t) + W_{M_2}(z(t)) * a_2 x_1(t) \quad (4)$

where $W_{M_1}(z(t))\ \&\ W_{M_2}(z(t))$ are the respective membership functions

by (4) we have

$$z(t) = (W_{M_1}(z(t)) * a_1 + W_{M_2}(z(t)) * a_2)\sin^{-1} z(t)) \quad (5)$$

$$W_{M_1}(z(t)) + W_{M_2}(z(t)) = 1 \quad (6)$$

From equation (5) & (6) we have:

$$W_{M_1}(z(t)) = \begin{cases} \frac{z(t) - a_2 \sin^{-1} z(t)}{(a_1 - a_2)\sin^{-1} z(t)}; & z(t) \neq 0 \\ 1 \quad ; & z(t) = 0 \end{cases}$$

$$W_{M_2}(z(t)) = 1 - W_{M_1}(z(t))$$

The membership function is shown below:

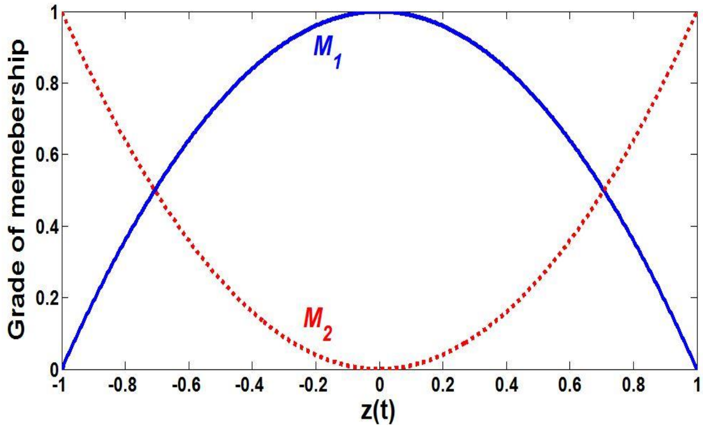

Fig. 3 Membership function

The model can be rewritten by introducing matrix representation as follows:

$$\dot{\boldsymbol{x}}(t) = (A_i + \Delta A_i)\boldsymbol{x}(t) + (B_i + \Delta B_i)\boldsymbol{u}(t); \quad (7)$$

for $i = 1,2$
where $\boldsymbol{x}(t) = [x_1(t) \quad x_2(t)]^T$

Model rule i: for $i = 1,2$

Rule 1: IF $z(t)$ *is* M_1

THEN $\dot{\boldsymbol{x}}(t) = (A_1 + \Delta A_1)\boldsymbol{x}(t) + (B_1 + \Delta B_1)\boldsymbol{u}(t)$

Rule 2: IF $z(t)$ *is* M_2

THEN $\dot{\boldsymbol{x}}(t) = (A_2 + \Delta A_2)\boldsymbol{x}(t) + (B_2 + \Delta B_2)\boldsymbol{u}(t)$

here $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ k_1 a_1 & 0 & k_2 \\ 0 & k_3 & k_4 \end{bmatrix}$;

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ k_1 a_2 & 0 & k_2 \\ 0 & k_3 & k_4 \end{bmatrix}$$

$$\Delta A_1 = \begin{bmatrix} 0 & 0 & 0 \\ \Delta k_1 a_1 & 0 & \Delta k_2 \\ 0 & \Delta k_3 & \Delta k_4 \end{bmatrix};$$

$$\Delta A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \Delta k_1 a_2 & 0 & \Delta k_2 \\ 0 & \Delta k_3 & \Delta k_4 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ k_5 \end{bmatrix};\ \Delta B_1 = \Delta B_2 = \begin{bmatrix} 0 \\ 0 \\ \Delta k_5 \end{bmatrix};$$

where

$k_1 = 9.8;\ k_2 = 1;\ k_3 = -10;\ k_4 = -10;\ k_5 = 10;\ a_1 = 1\ \&\ a_2 = \frac{2}{\pi}$

&

assume

$\Delta k_1 = \Delta k_3 = \Delta k_4 = \Delta k_5 = 0.02\ \&\ \Delta k_2 = 0.01.$

Given $\boldsymbol{x(t)}$ *and* $\boldsymbol{u(t)}$, *the model dynamics is inferred as follows:*

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \sum_{i=1}^{2} \mu_i\left(z(t)\right)\{(A_i + \Delta A_i)\boldsymbol{x}(\boldsymbol{t}) + (B_i + \Delta B_i)\boldsymbol{u}(\boldsymbol{t})\} \quad (8)$$

Where $\mu_i\left(z(t)\right) = \frac{w_i(z(t))}{\sum_{k=1}^{2} w_k(z(t))}$ &

$$w_i\left(z(t)\right) = \prod_{j=1}^{2} M_{ij}\left(z_j(t)\right);$$

$M_{ij}\left(z_j(t)\right)$ is the degree membership function $of\ z_j(t)\ in\ M_{ij}$.

Here $w_i\left(z(t)\right) \geq 0, for\ i = 1,2$
and $\sum_{k=1}^{2} w_i\left(z(t)\right) > 0$ for all t.

Therefore, $\mu_i\left(z(t)\right) \geq 0$
for $i = 1,2$ and $\sum_{i=1}^{2} \mu_i\left(z(t)\right) = 1$

2.2 State Feedback Fuzzy Controller

Control Rule 1:
IF $z(t)\ is\ M_1$
$THEN\ \boldsymbol{u}(\boldsymbol{t}) = -F_1\boldsymbol{x}(\boldsymbol{t})$

Control Rule 2:
IF $z(t)\ is\ M_2$
$THEN\ \boldsymbol{u}(\boldsymbol{t}) = -F_2\boldsymbol{x}(\boldsymbol{t})$

The fuzzy state feedback controller output is given by

$$\boldsymbol{u}(\boldsymbol{t}) = -\sum_{i=1}^{2} \mu_i\left(z(t)\right)F_i\boldsymbol{x}(\boldsymbol{t}) \quad (9)$$

where $\boldsymbol{x}(\boldsymbol{t}) = [x_1(t)x_2(t)]^{\boldsymbol{T}}$;

$$\mu_i\left(z(t)\right) = \frac{w_i(z(t))}{\sum_{k=1}^{2} w_k(z(t))};$$

3. STABILITY ANALYSIS

A sufficient condition, derived by H. K. Lam, F. H. F. Leung, and P. K. S. Tam in [9], which gives the stability of an uncertain fuzzy control system, is obtained in term of Lyapunov.

Theorem 1: The uncertain fuzzy control system (8) is guaranteed to be stable if there exists a common positive definite symmetric matrix $P \in R^{nxn}$ and a nonzero positive α such that following condition are satisfied:

1. $Q_m > 0$
2. $\sum_{k=1}^{r} Q_{ik} > 0$ for all i
3. $\alpha > 0$

where, $Q_m = -G_m^T P - PG_m$, G_m is to be chosen and it is a stable matrix.

$$Q_{ik} = -(G_{ik} - G_m)^T P - P(G_{ik} - G_m);$$

$$G_{ij} = A_i - B_i F_j;$$

$$\alpha = \lambda_{min}(Q_m) - \max\left\|\Delta Q_{ij}\right\|_{max}$$

where, $\max\left\|\Delta Q_{ij}\right\|_{max} \geq \left\|\Delta Q_{ij}\right\|$ for all i and j

$$\Delta Q_{ij} = \Delta G_{ij}{}^T P + P\Delta G_{ij};\quad \Delta G_{ij} = \Delta A_i - \Delta B_i F_j$$

4. RESULTS

All the calculations and simulation is performed in **Matlab 2016 version.** The feedback gains of Fuzzy state feedback controller are obtained by method of pole placement as given below:

$$\boldsymbol{F_1} = [\mathbf{20.12\quad 5.58\quad 0.40}]$$

$$\boldsymbol{F_2} = [\mathbf{15.1343\quad 5.2239\quad 0.40}]$$

The fuzzy controller output is obtained as:

$$\boldsymbol{u}(\boldsymbol{t}) = -[19.5016\ \ 5.5358\ \ 0.4000]\boldsymbol{x}(\boldsymbol{t})$$

The stable state responses of the uncertain fuzzy system are shown in figure below:

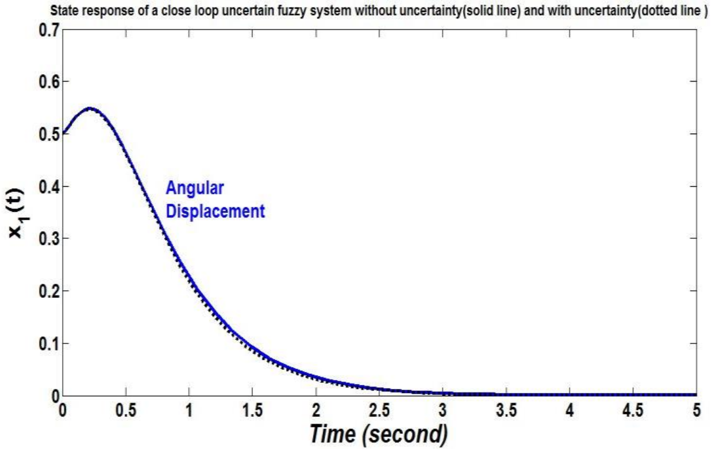

Fig. 4 State response of $x_1(t)$ of inverted pendulum controlled by a DC motor with parameter uncertainty (dotted line) and without parameter uncertainty (solid line) under initial condition $x(0) = [0.5 \;\; 0.2 \;\; 0]^T$

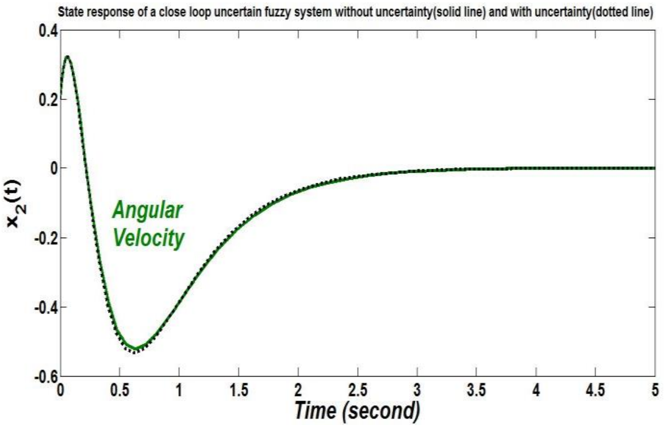

Fig. 5 State response of $x_2(t)$ of inverted pendulum controlled by a DC motor with parameter uncertainty (dotted line) and without parameter uncertainty (solid line) under initial condition $x(0) = [0.5 \;\; 0.2 \;\; 0]^T$

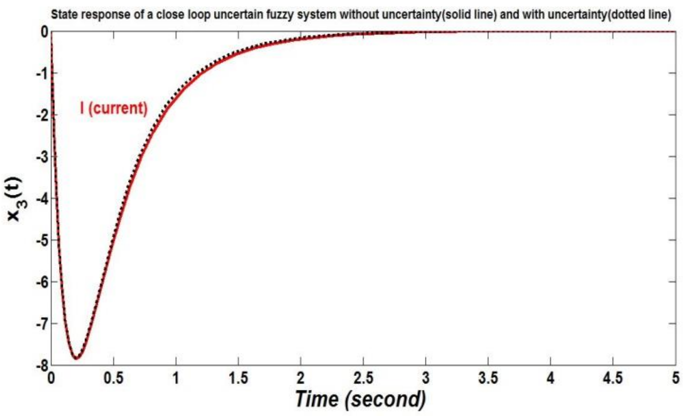

Fig. 6 State response of $x_3(t)$ of inverted pendulum controlled by a DC motor with parameter uncertainty (dotted line) and without parameter uncertainty (solid line) under initial condition $x(0) = [0.5 \;\; 0.2 \;\; 0]^T$

Stability analysis result:

$$G_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ -201.20 & -65.80 & -14 \end{bmatrix};$$

$$G_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ -151.3432 & -62.2388 & -14 \end{bmatrix};$$

$$G_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 6.2388 & 0 & 1 \\ -201.20 & -65.80 & -14 \end{bmatrix} \&$$

$$G_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 6.2388 & 0 & 1 \\ -151.3432 & -62.2388 & -14 \end{bmatrix}$$

To check the stability, we choose

$$G_m = \begin{bmatrix} 19.6304 & 8.9228 & -32.9335 \\ -47.1809 & -22.6625 & 12.7921 \\ -42.7112 & -38.8271 & -23.4440 \end{bmatrix}$$

which is stable matrix

By solving the LMIs as given below

1. $-G_m^T P - P G_m > 0$
2. $-\left(G_{ij} - G_m\right)^T P - P\left(G_{ij} - G_m\right) > 0 \; for \;\; i,j = 1,2$

where $G_{ij} = A_i - B_i F_j$; $\Delta Q_{ij} = \Delta G_{ij}{}^T P + P \Delta G_{ij}$;

$\Delta G_{ij} = \Delta A_i - \Delta B_i F_j$;

$\Delta Q_{ij} = \Delta G_{ij}{}^T P + P \Delta G_{ij}$

We have $\boldsymbol{P} = \begin{bmatrix} -0.4206 & -0.2509 & 0.1764 \\ -0.2509 & -0.2330 & 0.1274 \\ 0.1764 & 0.1274 & -0.0507 \end{bmatrix}$

Under this $\boldsymbol{P}$

we get $Q_m = \begin{bmatrix} 7.9062 & 4.2898 & -6.1242 \\ 4.2895 & 3.8099 & -2.9510 \\ -6.1242 & -2.9510 & 5.9823 \end{bmatrix}$

$\lambda_{min}(Q_m) = \mathbf{0.4960}$

Now $\left\|\Delta Q_{ij}\right\|$ is obtained for $i \,\&\, j = 1,2$ as summarize in

Table-1 below:

Table -1: l_2 Norm of the uncertain matrices

i,j	$\left\|\Delta Q_{ij}\right\|$

$i = 1, j = 1$	**0.0992**
$i = 2, j = 1$	0.0789
$i = 1, j = 2$	0.0947
$i = 2, j = 2$	0.0745

From table-1 we get $\|\Delta Q_{ij}\|_{max}$ =**0.0992**

$\alpha = \lambda_{min}(Q_m) - \|\Delta Q_{ij}\|_{max}$ = **0.3969** Since α is nonzero positive integer. This proves the given uncertain fuzzy system is stable.

5. CONCLUSIONS

The uncertain fuzzy model of the inverted pendulum controlled by a DC motor is derived and stability of fuzzy controller is checked with the Lyapunov stability. The fuzzy controller is simple design by the pole placement approach which is simple classical approach and the performance of the fuzzy control system has been checked with and without uncertainty. The final results shows that the approximate linear system is stable and it is out performed well in the presence of parameter uncertainty.

REFERENCES

[1] M. Gopal, *Digital Control and State Variable Methods*, 2nd ed.Tata McGraw-Hill, 2003.

[2] R. C. Dorf, R. H. Bishop, *Modern Control Systems*, 13 ed. Pearson Education, 2016.

[3] X.-J. Zeng and M. G. Singh, "Approximation theory of fuzzy systems—SISO case," *IEEE Trans. Fuzzy Syst.*, vol. 2, no. 2, pp. 162–176, May 1994.

[4] X.-J. Zeng and M. G. Singh, "Approximation accuracy analysis of fuzzy systems as function approximators," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1,pp. 44–63, Feb. 1996.

[5] G. Chen and T. Tat Pham, *Introduction to Fuzzy Sets, Fuzzy Logic and Fuzzy Control Systems*. CRC Press, Boca Raton, London, New York, Washington, D.C, 2001.

[6] K. M. Passino and S. Yurkovich, *Fuzzy Control*, Addison Wesley Longman, 1998.

[7] G. Feng, "A survey on analysis and design of model based fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 14, pp. 676-697, Oct 2006.

[8] S. Kawamoto, "Nonlinear Control and Rigorous Analysis Based On Fuzzy System for Interved Pendulum," The Fifth IEEE International Conference on Fuzzy Systems, vol. 2,, pp. 1429-1432, Sep. 1996.

[9] H. K. Lam, F. H. F. Leung, and P. K. S. Tam, "Stability and Robustness Analysis and Gain Design for Fuzzy Control Systems Subject to Parameter Uncertainties," The Ninth IEEE International Conference on Fuzzy Systems, 2000.

[10] H. Ohtake, K. Tanaka, H.O.Wang, "Fuzzy Modeling via Sector Nonlinearity Concept," Joint 9th IFSA World Congress and 20th NAFIPS International Conference, 2001.

BIOGRAPHIES

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