Solution of Linear Differential Equations with constant Coefficients

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Abstract - *This paper consist methods of finding the solution of linear differential equations with constant coefficients. Examples have been included to understand the different methods of finding the Particular solution for different functions given the RHS of LDE with constant coefficients, and Complementary functions for the different roots of Equations.*

Key Words: Complementary Function, Particular Integral, Complete solution, Rationalization, Auxiliary equation.

1. INTRODUCTION

Various systems of an engineering field such as oscillation of mechanical system, electrical systems, bending of beams, conduction of heat etc. can be expressed as differential equations. So it is necessary to study of the differential equations, in all respect as analysis, behavior and solution. In this introductory section, we discuss definitions, Rules of finding the Solution of linear differential equations with constant coefficient. Note that the proofs in this section are omitted, however if the reader is so inclined, the details are given in many standard texts on Differential Equations.

1.1 Linear Differential Equations:

A) Definition: A differential equation of the for

$$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots\ldots + A_n y = X \quad \ldots\ldots\ldots (1)$$

where A1, A2,…….. An and X are function of x or constants, is called A Linear Differential Equations with constant coefficient.

Put $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ……… $D^n = \frac{d^n}{dx^n}$ in equation (1)

In terms of D- notations (1) can be written as

f(D)y = X or

$(D_n y + A_1 D_{n-1} y + A_2 D_n\text{-}2y + \ldots\ldots\ldots + A_n y) = X$ …… (2)

B) Auxiliary Equation:

The equation f (D) = 0 is called auxiliary equation of equation (1) and (2)

C) Complementary Function:

It is the solution of equation F(D)y = X obtained by putting F(D)y = 0.

D) Particular Integral:

It is the solution of F(D)y = X which satisfies the given equation.

E) Complete Solution:

The Complete solution of equation (1) or (2) is given by

y= Complementary function (C.F.)+ Particular integral(P.I)

2. Methods of Finding Complementary Function (C.F):

Step I: find auxiliary equation (A.E.) f(m)=0 by writing D=m in f(D) of equation (2).

Step II: find the roots of the A.E. i.e. values of m. Let the roots are $m_1, m_2, \ldots\ldots, m_n$.

Step III: required C.F. is obtained as per the roots stated below:

Table1: **Rules of finding C.F**

Roots of A.E.	Complementary function (C.F.)
All roots $m_1, m_2, \ldots\ldots, m_n$ are real and distinct.	$C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots\ldots + C_n e^{m_n x}$
$m_1 = m_2$, but other roots are real and distinct.	$(C_1 + C_2 x)\, e^{m_1 x} + C_3 e^{m_2 x} + \ldots\ldots + C_n e^{m_n x}$
If roots are imaginary ($\alpha \pm \beta$) (say)	$e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$
If roots are imaginary and repeated twice	$e^{\alpha x}[(C_1 + C_2 x) \cos\beta x + (C_1 + C_2 x) \sin\beta x)$

1 Solved Examples:

Example 1: Solve (D^2- 3D - 4)y = 0.

Solution: Here Auxiliary equation is

(D^2- 3D - 4) = 0.

Or (D-4).(D+1) = 0

i.e., D = 4,-1

Hence roots are 4 and -1, real and different. Therefore C.F. is

$$y = c_1 e^{4x} + c_2 e^{-x}$$

Example 2: Solve (D^3 - 8)y = 0.

Solution: Here Auxiliary equation is

(D^3 - 8) = 0.

Or (D - 2). (D2 + 2D + 4) = 0

Or D = 2, D = -1± i$\sqrt{3}$

Hence roots are 2, and -1± i$\sqrt{3}$, one is real and other is a pair of imaginary. Therefore C.F. is

$$y = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$$

3. Short Methods for Finding Particular integral (P.I.):

If X ≠ 0, in equation (1) then $\text{P.I.} = \frac{1}{f(D)} X.$

Following are the methods for finding particular integral:

Table-2: **Rules for finding P.I.**

Types of function	What to do	Corresponding P.I.
X = e^{ax}	Put D = a in f(D)	$\frac{1}{f(D)} e^{ax}$, provided f(a) ≠ 0. If f(a) = 0 then (D-a) is one of the factor of f(D).This factor is solved by using the formula $\frac{1}{(D-a)} X = e^{ax} \int e^{-ax} X dx$. And rest is solve by above method given here.
X = x^m	take [f(D)]-1x^m	Expand [f(D)]-1 using binomial expansions and if (D-a) remains in the denominator then take rationalization of denominator. And treat D in the numerator as derivative of the corresponding function.
X = $e^{ax}v$	First operate e^{ax} on $\frac{1}{f(D)}$ then oper	$e^{ax} \frac{1}{f(D+a)} v$, then solve for v by above method.
X = sin ax (or cos ax)	Put D2= -a2 in f(D)	$\frac{1}{f(-a^2)}$ sin ax (or cos ax) , provided $\frac{1}{f(-a^2)} \neq 0$or otherwise use following formula: $\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$ or $\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$.

3.1 Solved Examples:

Example 1: Solve (D^2+ 4D+3)y = e^{-2x}.

Solution: Here Auxiliary equation is

(D^2+ 4D+3) =0

Or (D+3) (D+1) = 0

Or D = -3,-1

Hence roots are -3 and -1, real and different. Therefore C.F. is $\text{C.F} = C_1 e^{4x} + C_2 e^{-x}$

Now to find P.I:

$$\text{P.I} = \frac{1}{f(D)} X$$

$$= \frac{1}{(D2+ 4D+3)} e^{-2x}$$

Here X = e^{ax} therefore put D = a = -2

$$= \frac{1}{[(-2)^2+4(-2)+3]} e^{-2x}$$

$$= -e^{-2x}$$

Therefore, $\text{P.I} = -\mathbf{e}^{-2x}$

Hence the general solution is y = C.F.+ P.I.

i.e $y = C_1e^{4x} + C_2e^{-x} - e^{-2x}$

Example 2: Solve $(D^2 - 4)y = \boldsymbol{cos^2 x}$.

Solution: Here Auxiliary equation is

$(D^2 - 4) = 0.$

Or $(D-2)(D+2) = 0$

Or $D = 2, -2$

Hence roots are 2 and −2, real and different. Therefore C.F. is

$C.F = C_1\boldsymbol{e^{2x}} + C_2\boldsymbol{e^{-2x}}$

Now to find P.I:

$$P.I = \frac{1}{f(D)} X$$

$$= \frac{1}{(D^2-4)} \cos^2 x.$$

We know that $\cos^2 x = \frac{1+\cos 2x}{2}$ therefore,

$$= \frac{1}{(D^2-4)} \left[\frac{1+\cos 2x}{2}\right]$$

$$= \frac{1}{2} \frac{1}{(D^2-4)} (1+\cos 2x)$$

$$= \frac{1}{2} \frac{1}{(D^2-4)} (e^{0x}+\cos 2x)$$

$$= \frac{1}{2} \frac{1}{(D^2-4)} e^{0x} + \frac{1}{2} \frac{1}{(D2-4)} \cos 2x$$

$$= \frac{1}{2} \frac{1}{(0^2-4)} e^{0x} + \frac{1}{2} \frac{1}{(-2^2-4)} \cos 2x$$

$$= \frac{1}{-8} e^{0x} + \frac{1}{2} \frac{1}{-8} \cos 2x$$

$$P.I = \frac{1}{-8} + \frac{1}{-16} \cos 2x$$

Hence the general solution is y = C.F.+ P.I.

Therefore $y = C_1e^{2x} + C_2e^{-2x} + \frac{1}{-8} + \frac{1}{-16} \cos 2x$

Conclusion:

In this paper for the sack of convenience to students a brief summary related to the topic linear differential equations with constant coefficient, its kind and method to find the solutions in a simple manner is given. Some examples were given to show the effectiveness of a new method. It may be concluded that this technique is very dominant and capable in finding solutions. I hope that future students will also get the chance to work on such interesting problems.

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