A TERNARY QUADRATIC DIOPHANTINE EQUATION $39x^2 + 72xy - 39y^2 = 246z^2$

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Abstract: *The Homogeneous Ternary Quadratic Diophantine Equation is given by* $39x^2 + 72xy - 39y^2 = 246z^2$ *and analyzed for its patterns of non-zero distinct integer solutions. A few interesting relations among the solutions and special polygonal, pyramided, Mersenne, carol and Gnomonic and Pronic numbers are presented. Introducing the linear transformation x=u+v, y=u-v and employing the method of factorization, different patterns of non-zero distinct integer solutions to the above equations are obtained.*

Keywords: Homogeneous Quadratic, Ternary Quadratic, Integer solutions, polygonal number and pyramidal number, Mersenne number.

1. INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research by reason of their variety [1, 2]. In particular, one may refer [3, 18] for finding integer points on the some specific three dimensional surface. This communication concerns with yet another ternary quadratic Diophantine equation $39x^2 + 72xy - 39y^2 = 246z^2$ representing cone for determining its infinitely many integer solutions.

1.1 Notations Used:

1. $t_{m,n}$ =Polygonal number of rank 'n' with sides' m
2. P_n^m= Pyramidal number of rank 'n' sides m
3. Pr_n= Pronic number of rank 'n'
4. g_n = Gnomonic number
5. $car\ I_n$= Carol number
6. Mer_n= Mersenne number
7. ky_n= Kynea number

2. METHOD OF ANALYSIS

Consider the equation

$$39x^2 + 72xy - 39y^2 = 246z^2 \quad (1)$$

The transformed equation of (1) after using the linear transformations

$$x = u + v\,, y = u - v \quad (2)$$

$$(u \neq v \neq 0) \text{ is } 25u^2 + v^2 = 41z^2 \quad (3)$$

The above equation is solved through different methods and using (2), different patterns of integer solutions to (1) are obtained.

2.1 PATTERN

Write 41 as $41 = (4 + 5i)(4 - 5i)$ (4)

Assume that $z = 25a^2 + b^2$ where a, b>0 (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$$(5u + iv) = (4 + 5i)(5a + ib)^2 \quad (6)$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = \tfrac{1}{5}(100a^2 - 4b^2 - 5ab)$$

$$v = v(a,b) = 125a^2 - 5b^2 + 40ab \quad (7)$$

Substituting a=A and b=5B in (7) and (5), we get

$$u = u(A,B) = (20A^2 - 20B^2 - 5AB)$$

$$v = v(a,b) = 125A^2 - 125B^2 + 200AB$$

$$z = z(A,B) = 25A^2 + 25B^2 \quad (8)$$

Substituting the above u and v from (8) in (2), the values of x and y are given by

$$x = x(A,B) = 145A^2 - 145B^2 + 195AB$$

$$y = y(A,B) = 105B^2 - 105A^2 - 205AB \quad (9)$$

Thus (9) and (8) represent non-zero distinct integral solutions of equation (1) in two parameters.

PROPERTIES:

(1) $x(2^n, 1) = 145Mer_{2n} + 195Mer_n + 195$

(2) $x(2^n, 1) = 145ky_n - 95Mer_n - 95$

(3) $x(2^n, 1) = 145car\ I_n + 485Mer_n + 485$

(4) The number $z(2A, 2A)$ is a perfect square.

(5)$x(2a, a) + 3y(2a, a)$, a nasty number

2.2 PATTERN

In (3), $25u^2 + v^2 = 41z^2$

Consider the linear transformation

$z = X - 25T$ and $u = X - 41T$ (10)

Substituting (10) in (3), we get

$v^2 = 16(X^2 - 1025T^2)$ (11)

Write $v = 4V$ (12)

Substituting (12) in (11),

We get $V^2 = (X^2 - 1025T^2)$ (13)

The corresponding solution of (13) is

$T = 2ab$

$V = 1025a^2 - b^2$

$X = 1025a^2 + b^2$ (14)

Substituting (14) in (10) and (12), we get

$z = z(a, b) = 1025a^2 + b^2 - 50ab$

$u = u(a, b) = 1025a^2 + b^2 - 82ab$

$v = v(a, b) = 4(1025a^2 - b^2)$ (15)

Substituting (15) in (2), we get

$x = x(a, b) = 5125a^2 - 3b^2 - 82ab$

$y = y(a, b) = 5b^2 - 3075a^2 - 82ab$

$z = z(a, b) = 1025a^2 + b^2 - 50ab$ (16)

Thus (16) represents nonzero distinct integer solutions of (1) in two parameters a, b.

PROPERTIES:

(1)$3x(1, 2^n) + 5y(1, 2^n) = 16ky_n - 16Mer_{n+1} - 656Mer_n - 656$.

(2) $x(a, a) + y(a, a) - 1888t_{4,a} = 0$

(3) $2z(a, a) - x(a, a) - y(a, a) - 64t_{4,a} = 0$

(4) $x(2^n, 2^n) - 5125Mer_{2n} + 3Mer_n - 82Mer_{n+1} + 82ky_n - 5122 = 0$.

(5) $z(2^n, 2^m) - 1025Mer_{2n} - Mer_{2m} + 50Mer_{m+n} - 976 = 0$

2.3 PATTERN

Equation (3) can also be expressed as

$(5u + 5z)(5u - 5z) = (4z + v)(4z - v)$ (17)

in the form of ratio as

$$\frac{(5u + 5z)}{(4z + v)} = \frac{(4z - v)}{(5u - 5z)} = \frac{\mathrm{A}}{\mathrm{B}} \quad (18)$$

Where $B \neq 0$.

This is equivalent to the following system of equations

$-\mathrm{vA} + 5\mathrm{Bu} + (5\mathrm{B} - 4\mathrm{A})\mathrm{z} = 0$ (19)

$-5\mathrm{uA} - \mathrm{Bv} + (4\mathrm{B} + 5\mathrm{A})\mathrm{z} = 0$ (20)

On employing the method of cross multiplication, we get

$\mathrm{u} = -5\mathrm{B}^2 + 5\mathrm{A}^2 + 8AB$
$v = 20\mathrm{B}^2 - 20\mathrm{A}^2 + 50AB$ (21)

$\mathrm{z} = 5\mathrm{A}^2 + 5\mathrm{B}^2$ (22)

Substituting the values of u and v from (21) in (2), we get

$\mathrm{x} = \mathrm{x}(\mathrm{A}, \mathrm{B}) = 15\mathrm{B}^2 - 15\mathrm{A}^2 + 58AB$

$y = y(A, B) = 25A^2 - 25B^2 - 42AB$ (23)

Thus equations (22) and (23) represent the non-zero distinct integer solutions of equation (1) in two parameters A and B.

PROPERTIES

(1) $x(a, a) + y(a, a) = 16A^2$, a nasty number

(2)$x(a, a) - y(a, a)$, a perfect square

(3) $x(1, 2^n) - 15(ky_n) + 15Mer_{n+1} - 58Mer_n - 43 = 0$.

(4) $y(2^n, 1) - 25(car\ I_n) - 25Mer_{n+1} + 42Mer_n + 17 = 0$

(5)$z(A, A) - 10t_{4,A} = 0$

(6) $x(A, A + 1) + y(A, A + 1) - 16P_A + 10g_A + 30 = 0$

(7) $x(2^n, 2^m) + y(2^n, 2^m) - 10Mer_{2n} + 10Mer_{2m} - 16Mer_{m+n} - 16 = 0$

2.4 PATTERN

From (18) $\frac{(5u+5z)}{(4z-v)} = \frac{(4z+v)}{(5u-5z)} = \frac{\mathrm{A}}{\mathrm{B}}$ (24)

Where $B \neq 0$.

This is equivalent to the following system of equations

$vA + 5Bu + (5B - 4A)z = 0$ (25)

$-5uA + Bv + (4B + 5A)z = 0$ (26)

On employing the method of cross multiplication, we get

$u = 5A^2 - 5B^2 + 8AB$

$v = 20A^2 - 20B^2 - 50AB$ (27)

$z = 5A^2 + 5B^2$ (28)

Substituting the values of u and v from (27) in (2), we get

$x = x(A, B) = 25A^2 - 25B^2 - 42AB$

$y = y(A, B) = 15B^2 - 15A^2 + 58AB$ (29)

Thus equations (28) and (29) represent the non-zero distinct integer solutions of equation (1) in two parameters A and B.

PROPERTIES

(1)$x(2^n, 2^n) + 42Mer_{2n} + 42 = 0$

(2) $y(2^n, 2^n) - 58Mer_{2n} - 58 = 0$

(3)$x(2^n, 1) + y(1, 2^n) - 25ky_n + 15Mer_{n+1} - 54Mer_n - 39 = 0$

(4) $x(2^n, 2^m) + y(2^n, 2^m) - 10ky_n + 10ky_m - 16Mer_{m+n} - 16 = 0.$

(5) $z(2^n, 1) - 5ky_n + 5Mer_{n+1} - 5 = 0.$

2.5 PATTERN

Equation (13) can be written as

$$X^2 - V^2 = 1025T^2$$

Hence $(X + V) = 41T$ (30)

$(X - V) = 25T$ (31)

Solving (30) and (31), we get

$X = 33T$ and $V = 8T$ (32)

Substituting (32) in (12), we get

$v = 32T$ (33)

For T=A, we have $v = 32A$

Substituting the value of v from (33) in (10), we get $z = 8T$ and $u = -8T$

Thus for T=A, $z = z(A) = 8A$

$u = u(A) = -8A$ (34)

Substituting the values of u and z from (34) in (2), we get

$x = x(A) = 24A$

$y = y(A) = -40A$

$z = z(A) = 8A$, from (34)

Thus the above values of x, y and z represent the non-zero distinct integer solutions of (1) in the parameter A.

PROPERTIES

(1) $x(A^2) + y(A^2) + z(A^2)$, a nasty number.

(2) $x(A^2) + y(A^2)$,a perfect square.

(3) $x(A^2) + 3z(A^2) - t_{4,8A} = 0$

(4)$x(2^n) + y(2^n) + 3z(2^n) - 40Mer_n - 40 = 0$

(5)$y(A^2)$, a nasty number.

(6)$\frac{1}{3}[x(A^2) - y(A^2) + z(A^2)]$, a nasty number.

2.6 PATTERN

Equation (3) can also be expressed in the form of ratio as

$\frac{41(u+z)}{(4u-v)} = \frac{(4u+v)}{(u-z)} = \frac{A}{B}$ (35)

Where $B \neq 0$.

This is equivalent to the following system of equations

$vA + (41B - 4A)u + (41B)z = 0$ (36)

$(4B - A)u + Bv + Az = 0$ (37)

On employing the method of cross multiplication, we get

$u = u(A, B) = A^2 - 41B^2$

$v = v(A, B) = 4A^2 + 164B^2 - 82AB$ (38)

$z = z(A, B) = A^2 - 8AB + 41B^2$ (39)

Substituting the values of u and v from (38) in (2), we get

$x = x(A, B) = 5A^2 + 123B^2 - 82AB$

$y = y(A, B) = 82AB - 3A^2 - 205B^2$ (40)

Thus equations (39) and (40) represent the non-zero distinct integer solutions of equation (1) in two parameters A and B.

PROPERTIES

(1)$x(2^n,2^m)+y(2^n,2^m)-2Mer_{2n}+82Mer_{2m}+80=0$.

(2)$\frac{1}{2}[3x(a,a)+y(a,a)]$, a nasty number

(3) $z(2^n,2^n)-46ky_n+46Mer_{n+1}=0$

(4)$x(a,a)+y(a,a)\equiv 0 mod 20$

(5) $z(2^n,1)-Mer_{2n}-8Mer_n-50=0$

3. CONCLUSION

In this paper, I have presented different pattern of integer solutions to the ternary quadratic Diophantine equation $39x^2+72xy-39y^2=246z^2$ representing a cone. As these Diophantine equations are rich in variety, one may attempt to find integer solutions to other choices of equations along with suitable properties.

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BIOGRAPHY

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