ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $4x^2 - 7xy + 4y^2 + x + y + 1 = 19z^2$

Chikkavarapu Gnanendra Rao

Assistant Professor of mathematics, Department S&H, SSIET, Nuzvid, Andhra Pradesh, India

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Abstract: *The Non-Homogeneous Ternary Quadratic Diophantine Equation given by $4x^2 - 7xy + 4y^2 + x + y + 1 = 19z^2$ represents a cone and is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions and special polygonal and pyramided numbers are presented. On introducing the linear transformation x=u+v, y=u-v and employing the method of factorization, different patterns of non-zero distinct integer solutions to the above equation are obtained.*

Keywords: Non-Homogeneous Quadratic, Ternary Quadratic, Integer solutions, polygonal number and pyramidal number.

1. INTRODUCTION

The Ternary Quadratic Diophantine equations offer an unlimited field for research by reason of their variety [1, 2]. In particular, one may refer [3, 19] for finding integer points on the some specific three dimensional surface. This communication concerns with yet another ternary quadratic Diophantine equation $4x^2 - 7xy + 4y^2 + x + y + 1 = 19z^2$ representing cone for determining its infinitely many integer solutions.

1.1 Notations Used:

1. Polygonal number of rank 'n' with sides' m

$$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$$

2. Stella Octangular number of rank 'n'

$$SO_n = n(2n^2 - 1)$$

3. Pyramidal number of rank 'n' sides' m

$$P_n^m = \frac{n(n+1)}{6}[(n-1)n + (5-m)]$$

4. Pronic number of rank 'n'

$$Pr_n = n(n+1)$$

2. METHOD OF ANALYSIS

The Ternary Quadratic Diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$4x^2 - 7xy + 4y^2 + x + y + 1 = 19z^2 \quad (1)$$

Using the linear transformation $x = u + v$,

$$y = u - v \qquad (u \neq v \neq 0) \quad (2)$$

Equation (1) becomes

$$(u+1)^2 + 15v^2 = 19z^2 \quad (3)$$

Different patterns of solution of (1) are presented below

PATTERN-I:

Write $19 = (2 + i\sqrt{15})(2 - i\sqrt{15})$ (4)

Assume that $z = a^2 + 15b^2$ where a, b>0 are distinct integers. (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$$\left((u+1) + i\sqrt{15}v\right) = (2 + i\sqrt{15})(a + i\sqrt{15}b)^2 \quad (6)$$

Equating the real and imaginary parts, we get $u = u(a,b) = 2a^2 - 30b^2 - 30ab - 1$ $\quad v = v(a,b) = a^2 - 15b^2 + 4ab$

Substituting the above values of u and v in (2), the values of x and y are given by

$$x = x(a,b) = 3a^2 - 45b^2 - 26ab - 1$$

$$y = y(a,b) = a^2 - 15b^2 - 34ab - 1 \quad (7)$$

Thus equations (5) and (7) represent non-zero distinct integral solution of equation (1) in two parameters.

PROPERTIES:

(1) $z(a,a)$ is a perfect square.

(2) $x(a,b)+y(a,b)\equiv 0 mod 2$

(3) $x(1,b)+y(1,b)+120t_{3,a}=2$

(4) $x(a,a+1)-3y(a,a+1)-152t_{3,a}=2$

(5) $x(2a,2b)-y(2a,2b)\equiv 0 mod 8$

PATTERN-II:

Instead of equation (4), we write

$$19=\frac{(4+2i\sqrt{15})(4-2i\sqrt{15})}{4} \quad (8)$$

Write $1=\frac{(1+i\sqrt{15})(1-i\sqrt{15})}{16}$ (9)

Assume that $z=a^2+15b^2$ where a, b>0 are distinct integers. (10)

Equation (3) can be written as

$$(u+1)^2+15v^2=19z^2*1 \quad (11)$$

Using the equations (8) & (9) in (11) and applying the method of factorization, define

$$\left((u+1)+i\sqrt{15}v\right)$$

$$=\frac{1}{8}\left(2+i\sqrt{15}\right)\left(1+i\sqrt{15}\right)\left(a+i\sqrt{15}b\right)^2 \quad (12)$$

Equating the real and imaginary parts, we get

$$u=u(a,b)=\frac{1}{8}(390b^2-26a^2-180ab-8)$$

$v=v(a,b)=\frac{1}{8}(6a^2-90b^2-52ab)$ Substituting the above u and v in (2), the values of x and y are given by

$$x=x(a,b)=\frac{1}{8}(300b^2-20a^2-232ab-8)$$

$$y=y(a,b)=60b^2-4a^2-16ab-1 \quad (13)$$

For finding the integer solutions, we choose the values of a, b such that u and v are integers.

Replacing a by 2A and b by 2B, in the equations (10) and (13) we get

$$u=u(a,b)=u(2A,2B)$$

$$=(195B^2-13A^2-90AB-1)$$

$$v=v(a,b)=v(2A,2B)$$

$$=(3A^2-45B^2-26AB)$$

$$z=z(a,b)=z(2A,2B)=(4A^2+60B^2) \quad (14)$$

Substituting the values of u and v in equation (2) the values of x and y are given by

$$x=x(A,B)=150B^2-10A^2-116AB-1$$

$$y=y(A,B)=240B^2-16A^2-64AB-1$$

$$z=z(A,B)=4A^2+60B^2 \quad (15)$$

Thus the equation (15) represents non-zero distinct integral solutions of equation (1) in two parameters A, B.

PROPERTIES:

(1) $z(a,a)-t_{4,8a}=0$.

(2) $x(a,a)+y(a,a)+582t_{4,a}+2=0$

(3) $x(a,a+1)+y(a,a+1)\equiv 0 mod 2$

(4) $x(a,a)+y(a,a)+z(a,a)+518t_{4,a}+2=0$

(5) $y(2a,a)+z(2a,a)+724Pr_a-724a=0$

(6) $z(a,a)-64Pr_a\equiv 0 mod 64$

PATTERN-III:

Equation (3) can be written as

$$(u+1)^2=19z^2-15v^2 \quad (16)$$

Using the linear transformation

$$z=X+15T \text{ and } v=X+19T \quad (17)$$

From (16) and (17),

we get $X^2=285T^2+\left(\frac{u+1}{2}\right)^2$ (18)

Equation (18) is satisfied by

$$T(m,n)=2mn$$

$$u(m,n)=570m^2-2n^2-1$$

$$X(m,n)=285m^2+n^2 \quad (19)$$

Substituting the above values from (19) in (17) and using (2), the required integer solutions of (1) are given by

$x = x(m,n) = 855m^2 - n^2 + 38mn - 1$

$y = y(m,n) = 285m^2 - 3n^2 - 38mn - 1$

$z = z(m,n) = 285m^2 + n^2 + 30mn \qquad (20)$

Therefore the equation (20) represents non-zero distinct integral solutions of the equation (1) in two parameters.

PROPERTIES:

(1) $x(m,m) - y(m,m) - z(m,m) - 83t_{4,2m} = 0$

(2) $x(m,m) - y(m,m) + z(m,m) - 241t_{4,2m} = 0$

(3) $z(m,m) - 79t_{4,2m} = 0$

(4) $x(m,m+1) - y(m,m+1) - 570t_{4,m} - 2t_{4,m+1} - 72Pr_m = 0$

(5) $y(m,m) - z(m,m) + 72t_{4,m} + 1 = 0$

NOTE: In addition to (17) we may use the linear transformation $z = X - 15T$ and $v = X - 19T$ to get the different set of solutions as seen above.

PATTERN-IV

Consider (3) as

$(u+1)^2 - 4v^2 = 19(z^2 - v^2) \qquad (21)$

$$\frac{u+1+2v}{z+v} = \frac{19(z-v)}{u+1-2v} = \frac{A}{B}, B \neq 0$$

This is equivalent to the following two equations

$B(u+1) + v(2B - A) - zA = 0$ and

$A(u+1) + v(19B - 2A) - 19zB = 0$.

On employing the method of cross multiplication, we get

$u = u(A,B) = 38AB - 38B^2 - 2A^2 - 1$

$v = v(A,B) = 19B^2 - A^2 \qquad (22)$

$z = z(A,B) = 19B^2 + A^2 - 4AB \qquad (23)$

Substituting the values of u and v from (22) in (2), the non-zero distinct integral values of x, y are given by

$x = -3A^2 - 19B^2 + 38AB - 1$

$y = -A^2 - 57B^2 + 38AB - 1$

$z = 19B^2 - 4AB + A^2 \qquad (24)$

Thus the equation (24) represents non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES:

(1) $z(A,A) - t_{4,4A} = 0$

(2) $x(A,A) + y(A,A) + t_{4,2A} + 1 = 0$

(3) $x(A,A) - y(A,A) - t_{4,6A} + 1 = 0$

(4) $z(A,A+1) - 19t_{4,A+1} + 2{P_n}^2 - t_{4,A} = 0$

(5) $x(A,A) - y(A,A) - z(A,A) - 20t_{4,A} = 0$

NOTE: Equation (21) can also be expressed, in the form of ratio in three different ways that are given below.

PATTERN-V

$$\frac{u+1+2v}{19(z-v)} = \frac{(z+v)}{u+1-2v} = \frac{A}{B}, B \neq 0$$

This is equivalent to the following two equations

$B(u+1) + v(2B + 19A) - 19zA = 0$ and

$A(u+1) + v(-2A - B) - Bz = 0$

On employing the method of cross multiplication, we get

$u = u(A,B) = 38AB + 2B^2 + 38A^2 - 1$

$v = v(A,B) = 19A^2 - B^2 \qquad (25)$

$z = z(A,B) = B^2 + 19A^2 + 4AB \qquad (26)$

Substituting the values of u and v from (25) in (2), the non-zero distinct integral values of x, y are given by

$x = 57A^2 + B^2 + 38AB - 1$

$y = 19A^2 + 3B^2 + 38AB - 1$

$z = B^2 + 4AB + 19A^2 \qquad (27)$

Thus the equation (24) represents non-zero distinct integral solutions of (1) in two parameters.

PATTERN-VI

$$\frac{u+1-2v}{(z+v)}=\frac{19(z-v)}{u+1+2v}=\frac{A}{B}, B\neq 0$$

This is equivalent to the following two equations

$B(u+1)+v(-2B-A)-zA=0$ and

$$A(u+1)+v(19B+2A)-19zB=0$$

On employing the method of cross multiplication, we get

$u=u(A,B)=38AB+38B^2+2A^2-1$

$v=v(A,B)=19B^2-A^2$ (28)

$z=z(A,B)=19B^2+A^2+4AB$ (29)

Substituting the values of u and v from (28) and (29) in (2), the non-zero distinct integral values of x, y are given by

$x=A^2+57B^2+38AB-1$

$y=3A^2+19B^2+38AB-1$

$z=19B^2+4AB+A^2$ (30)

Thus the equation (30) represents non-zero distinct integral solutions of (1) in two parameters.

PATTERN-VII

$$\frac{u+1-2v}{19(z-v)}=\frac{(z+v)}{u+1+2v}=\frac{A}{B}, B\neq 0$$

This is equivalent to the following two equations

$B(u+1)+v(-2B+19A)-19zA=0$ and

$$A(u+1)+v(2A-B)-zB=0$$

On employing the method of cross multiplication, we get

$u=u(A,B)=38A^2-38AB+2B^2-1$

$v=v(A,B)=B^2-19A^2$ (31)

$z=z(A,B)=4AB-B^2-19A^2$ (32)

Substituting the values of u and v from (31) and (32) in (2), the non-zero distinct integral values of x, y are given by

$x=19A^2+3B^2-38AB-1$

$y=57A^2+B^2-38AB-1$

$z=4AB-B^2-19A^2$ (33)

Thus the equation (33) represents non-zero distinct integral solutions of (1) in two parameters.

3. CONCLUSION:

In this paper I have presented seven different patterns of non-zero distinct integer solutions of the non-homogeneous cone given by$4x^2-7xy+4y^2+x+y+1=19z^2$. To conclude, one may search for patterns of non-zero distinct integer solutions and their corresponding properties for other choices of ternary quadratic Diophantine equations.

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AUTO-BIOGRAPHY

Chikkavarapu Gnanendra Rao, (born on 31-1-1983 in Nuzvid, Andhra Pradesh) completed post-graduation in mathematics from P.B.Siddartha College of arts and science in Vijayawada in the year 2005. Presently he is working as assistant professor of mathematics in Sri Sarathi Institute of Engineering and Technology. He was qualified in GATE 2013 and APSET-2012.