International Research Journal of Engineering and Technology (IRJET) e-ISSN: 2395-0056
Volume: 04 Issue: 08 | Aug -2017 www.irjet.net p-ISSN: 2395-0072

The Existence of Maximal and Minimal Solution of Quadratic Integral Equation

Amany M. Moter

Lecture, Department of Computer Science, Education College, Kufa University, Najaf, Iraq

---***---

Abstract - *In this paper, we study existence of solution of quadratic integral equations*

$$x(t) = g(t,x) + x(t)\int_0^t k(t,s)f(t,s,x)ds,$$

$$t \in [0,1] \qquad (1)$$

by using Tychonoff fixed point theorem. Also existence maximal and minimal solution for equation (1).

Key words: quadratic integral equation, maximal and minimal solution, Tychonoff Fixed Point Theorem.

1. Introduction

In fields, physics and chemistry, they can be use quadratic integral equations (QIEs) in their applications, for examples: the theory of radiative transfer, traffic theory, kinetic theory of gases and neutron transport and in many other phenomena.

The paper ([1-10]) studied quadratic integral equations.
Thus, we study solvability of the following quadratic integral equation:

$$x(t) = g(t,x)$$

$$+x(t)\int_0^t k(t,s)f(t,s,x)ds, \quad t \in [0,1] \qquad (1)$$

2. Preliminaries

We need in our work the following fixed point theorems and definitions

Definition 1[11]: A set $S \subset R$ is said to be convex if, $\forall\lambda \in [0,1]$ and $\forall x, y \in S$, $\lambda x + (1-\lambda)y \in S$.

If $x, y \in R$ and $\lambda \in [0,1]$, then $\lambda x + (1-\lambda)y$ is said to be a convex combination of x and y.

Simply says that S is a convex set if any combination of every two elements of S is also in S.

Theorem 2 (Tychonoff Fixed Point Theorem) [12]: suppose B is a complete, locally convex linear space and S is a closed convex subset of B. Let a mapping $T: B \to B$ be continuous and $T(S) \subset S$. If the closure of $T(S)$ is compact, then T has a fixed point.

Theorem 3 (Arzelà-Ascoli Theorem) [13]: Let E be a compact metric space and $C(E)$ the Banach space of real or complex valued continuous functions normed by

$$\|f\| = \sup_{t\in E}|f(t)|$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that f_n is uniformly bounded and equicontinuous, then $\bar{A}$ is compact.

To prove the existence of continuous solution for quadratic integral equation(1), we let $I = [0,1]$, $L_1 = L_1[0,1]$ be the space of Lebesgue integrable function I and R be the set of real numbers.

3. Existence of solution

We study the existence of at least one solution of the integral equation (1) under the following assumptions:

(i)- $g: I \times R_+ \to R_+$ is continuous, and there a exist function

$m: R_+ \to R_+$ Such that $|g(t,x)| \leq m(|x|)$
(ii)- $k: [0,1] \times [0,1] \to R_+$ is continuous for two variables t and s such that:

$$|k(t,s)| \leq K \qquad \text{for all } t \in [0,1]$$

Where K is constant $(K > o)$.

(iii) $f: I \times I \times R_+ \to R_+$ is bounded function and satisfies Carathéodory condition, Also there exist continuous

function $a, b: R_+ \to R_+$ satisfying

$|f(t,s,x(s))| \leq a(t)b(s)$ for all $t, s \in [0,1]$ and $x \in R_+$

(iv) There exists a constant $c \in [0,1]$ such that

$$Ka(t)\int_0^t b(s)ds \leq c$$

And $A = \sup\{|f(t_2,s,x) - f(t_1,s,x)| : \forall t \in [0,1]\}$

Now we can formulate the main theorem

Theorem 4: if the assumptions (i) and (iii) are satisfied, then the quadratic integral equation of Volterra type has at least one solution $x \in C[0,1]$.

Proof: Let C be set of all continuous function on interval $[0,1]$ denoted by $C[0,1]$, it is a complete locally convex linear space that has been proved in [12], and define the set B by

$$B = \{x \in C: |x(t)| \leq r\}, t \in [0,1]$$

Where $r = m(r) + rq$

Clearly B is nonempty, bounded and closed, but we will prove that the set B convex.

Let $x_1, x_2 \in B$ and $\lambda \in [0,1]$, then we have

$$\|\lambda x_1 + (1-\lambda)x_2\| \leq \lambda\|x_1\| + (1-\lambda)\|x_2\|$$

$$\leq \lambda r + (1-\lambda)r$$

$$= \lambda r + r - \lambda$$

$$= r$$

Then $\lambda x_1 + (1-\lambda)x_2 \in B$, which means that B is convex set.

To show that $H: B \to B$, let $x \in B$, then

$$|x(t)| = \left| g(t,x) + x(t)\int_0^t k(t,s)f(t,s,x)ds \right|$$

$$\leq |g(t,x)| + |x(t)| \int_0^t |k(t,s)||f(t,s,x)|ds$$

$$\leq |g(t,x)| + |x(t)| \int_0^t |k(t,s)|a(t)b(s)ds$$

$$\leq m(r) + rKa(t)\int_0^t b(s)ds$$

$$\leq m(r) + rq$$

$$= r$$

This means that B is closed and by similar steps we can prove $HB \subset B$

Consider the operator H:

$$Hx(t) = g(t,x)$$

$$+x(t)\int_0^t k(t,s)f(t,s,x)ds, \qquad t \in [0,1]$$

$$|Hx(t)| =$$

$$\left| g(t,x) + x(t)\int_0^t k(t,s)f(t,s,x)ds \right|$$

$$\leq m(t) + rq$$

$$= r$$

Then $Hx \in B$ implies to $HB \subset B$

Now, Let $t_1, t_2 \in I, t_1 < t_2$ and $|t_2 - t_1| \leq \delta$, then

$$|Hx(t_2) - Hx(t_1)| =$$

$$\Big| g(t_2,x) + x(t_2)\int_0^{t_2} k(t_2,s)f(t_2,s,x)ds \; - g(t_1,x) +$$
$$x(t_1)\int_0^{t_1} k(t_1,s)f(t_1,s,x)ds \Big|$$

$$= \Big| g(t_2,x) + x(t_2)\int_0^{t_2} k(t_2,s)f(t_2,s,x)ds$$

$$-g(t_1,x) - x(t_2)\int_0^{t_2} k(t_2,s)f(t_2,s,x)ds$$

$$+g(t_1,x) + x(t_2)\int_0^{t_2} k(t_2,s)f(t_2,s,x)ds$$

$$-g(t_1,x) - x(t_1)\int_0^{t_2} k(t_2,s)f(t_2,s,x)ds$$

$$+g(t_1,x) + x(t_1)\int_0^{t_2} k(t_2,s)f(t_2,s,x)ds$$

$$-g(t_1,x) - x(t_1)\int_0^{t_1} k(t_2,s)f(t_2,s,x)ds$$

$+g(t_1,x)+x(t_1)\int_0^{t_1} k(t_2,s)f(t_2,s,x)ds$

$-g(t_1,x)-x(t_1)\int_0^{t_1} k(t_1,s)f(t_2,s,x)ds$

$+g(t_1,x)+x(t_1)\int_0^{t_1} k(t_1,s)f(t_2,s,x)ds$

$-g(t_1,x)-x(t_1)\int_0^{t_1} k(t_1,s)f(t_1,s,x)ds \Big|$

$\leq$
$|g(t_2,x)-g(t_1,x)|+$
$|x(t_2)-x(t_1)|\int_0^{t_2}|k(t_2,s)||f(t_2,s,x)|ds$

$+|x(t_1)|\int_{t_1}^{t_2}|k(t_2,s)||f(t_2,s,x)|ds$

$+|x(t_1)|\int_0^{t_1}|k(t_2,s)-k(t_1,s)||f(t_2,s,x)|ds$

$+|x(t_1)|\int_0^{t_1}|k(t_1,s)||f(t_2,s,x)-f(t_1,s,x)|ds$

We have

$\leq$
$\varepsilon_1+|x(t_2)-x(t_1)|Ka(t)\int_0^{t_2} b(s)+rK\int_{t_1}^{t_2}|f(t_2,s,x)|ds+$
$r\varepsilon_2 a(t)\int_0^{t_1} b(s)$

$+rKAt_1 \to 0$ as $t_2 \to t_1$.

This means that the function HB is equi-continuous on[0,1]. By using Arzela-Ascoli theorem, we can say that HB is compact.

Tychonoff fixed point theorem is satisfied all its conditions, then the operator H has at least one fixed point. This completes the proof.

4 Maximal and minimal solution

Definition 5: [14] let $q(t)$ be a solution of equation (1) then $q(t)$ is said to be a maximal solution of equation (1) if every solution of (1) on [0,1] satisfies the inequality

$$x(t) < q(t), \qquad t \in I$$

A minimal solution $p(t)$ can be defined in a similar way by reversing the above inequality i.e $x(t) > q(t),\ t \in I$

The following lemma important to prove the existence of maximal and minimal solution of equation (1).

Lemma 6: suppose that $f(t,x)$ satisfies the assumption (i) of theorem 1 and let $x(t), y(t)$ be continuous function on $[0,1]$ satisfying

$$x(t) \leq g(t,x)+x(t)\int_0^t k(t,s)f(t,s,x)ds$$

$$y(t) \geq g(t,x)+x(t)\int_0^t k(t,s)f(t,s,x)ds$$

And one of them is strict.

Let $f(t,x)$ is nondecreasing function in x then

$x(t) < y(t), \qquad t \in [0,1] \qquad (2)$

Proof: Let conclusion (2) be false, then there exists t_1 such that

$x(t_1) = y(t_1), \qquad t_1 > 0 \qquad (3)$

And

$x(t) < y(t), \qquad 0 < t < t_1, \qquad t, t_1 \in [0,1]$

From the monotonicity of f in , we get

$$x(t_1) \leq g(t_1,x)+x(t_1)\int_0^{t_1} k(t_1,s)f(t_1,s,x)ds$$

$$< g(t_1,y)+y(t_1)\int_0^{t_1} k(t_1,s)f(t_1,s,y)ds$$

That implies to

$$x(t_1) < y(t_1)$$

This is contradiction with (3), then

$$x(t) < y(t).$$

Next, we prove the existence maximal and minimal solution of quadratic integral equation (1). So, we have the next theorem.

Theorem: let all conditions of theorem 1 be satisfied and if $f(t,x)$ is nondecreasing functions in x, then there exist maximal and minimal solutions of equation (1).
Proof: for the existence of the maximal solution

let $\varepsilon > 0$ be given and

$$f_\varepsilon(t,s,x_\varepsilon) = f(t,s,x_\varepsilon)+\varepsilon,$$

$$g_\varepsilon(t,x_\varepsilon) = g(t,x_\varepsilon)+\varepsilon$$

From equation (1) we obtain that:

$x_\varepsilon(t) = (g(t, x_\varepsilon) + \varepsilon)$

$+x_\varepsilon(t) \int_0^t k(t,s)(f(t,s,x_\varepsilon) + \varepsilon)ds$

$= g_\varepsilon(t, x_\varepsilon) + x_\varepsilon(t) \int_0^t k(t,s) f_\varepsilon(t,s,x_\varepsilon)ds \quad (4)$

Clearly the functions $g_\varepsilon(t, x_\varepsilon)$ and $f_\varepsilon(t,s,x_\varepsilon)$ satisfy assumptions $(i), (iii)$ then equation (4) has a continuous solution on $C(I)$.

Let ε_1 and ε_2 be such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$ then

$x_{\varepsilon_2}(t) = g_{\varepsilon_2}(t, x_{\varepsilon_2}) + x_{\varepsilon_2}(t) \int_0^t k(t,s) f_{\varepsilon_2}(t,s,x_{\varepsilon_2})ds$

$= (g(t, x_{\varepsilon_2}) + \varepsilon_2) + x_{\varepsilon_2}(t) \int_0^t k(t,s)(f(t,s,x_{\varepsilon_2}) +$

$\varepsilon_2)ds \quad (5)$

Also

$$x_{\varepsilon_1}(t) = g_{\varepsilon_1}(t, x_{\varepsilon_1}) + x_{\varepsilon_1}(t) \int_0^t k(t,s) f_{\varepsilon_1}(t,s,x_{\varepsilon_1})ds$$

$= (g(t, x_{\varepsilon_1}) + \varepsilon_1)$

$$+x_{\varepsilon_1}(t) \int_0^t k(t,s)(f(t,s,x_{\varepsilon_1}) + \varepsilon_1)ds \qquad (6)$$

$x_{\varepsilon_1}(t) > (g(t, x_{\varepsilon_1}) + \varepsilon_2)$

$$+x_{\varepsilon_1}(t) \int_0^t k(t,s)(f(t,s,x_{\varepsilon_1}) + \varepsilon_2)ds \qquad (7)$$

Applying lemma 6 to (5) and (7) we have

$$x_{\varepsilon_2}(t) < x_{\varepsilon_1}(t), \qquad t \in [0,1]$$

According to the previous of the theorem 1, we conclude that equation (4) is equi-continuous and uniformly bounded, through it we use the Arzela-Ascoli theorem so, there exists a decreasing sequence ε_n such that $\varepsilon \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} x_{\varepsilon_n}(t)$ exists uniformly in I and we denote this limit by $q(t)$. From the continuity of the functions f_ε and g_ε in the second argument, we get

$$f_\varepsilon(t,s,x_\varepsilon) \to f(t,s,q(t)) \qquad \text{as } n \to \infty$$

$$g_\varepsilon(t,x_\varepsilon) \to g(t,q(t)) \qquad \text{as } n \to \infty$$

and

$(t) = \lim_{n\to\infty} x_{\varepsilon_n}(t) =$

$g(t, q(t)) + q(t) \int_0^t k(t,s) f(t,s,q(s))ds$

which implies that $q(t)$ is a solution of equation (1). Now, we can prove that $q(t)$ is the maximal solution of quadratic integral equation (1)

Let $x(t)$ be any solution of equation (1), then

$x(t) =$

$g(t,x) + x(t) \int_0^t k(t,s) f(t,s,x)ds \qquad (8)$

and

$x_\varepsilon(t) = (g(t, x_\varepsilon) + \varepsilon)$

$+x_\varepsilon(t) \int_0^t k(t,s)(f(t,s,x_\varepsilon) + \varepsilon)ds$

$x_\varepsilon(t) >$

$g(t,x_\varepsilon) + x_\varepsilon(t) \int_0^t k(t,s) f(t,s,x_\varepsilon)ds \qquad (9)$

by Lemma 6 and equations (8), (9) we get

$$x(t) < x_\varepsilon(t), \qquad t \in [0,1]$$

From the uniqueness of the maximal solution (see [14] and [15]), it is clear that $x_\varepsilon(t)$ tends to $q(t)$ uniformly in [0,1] as $\varepsilon \to 0$.

In the same manner we can prove the existence of the minimal solution.

4. Conclusion:

Equation (1) has a maximal and minimal solution after we proved the existence of at least one solution by using Tychonoff Fixed Point Theorem under 4 assumptions.

References

[1] J.Banaś, M. Lecko and W. G. El Sayed, "Existence Theorems of some quadratic Integral Equation". J. Math. Anal. Appl., 227(1998), 276-279.

[2] J.Banaś and A. Martianon, "Monotonic Solution of a quadratic Integral equation of

Volterra Type". Comput. Math. Apple., 47 (2004), 271-279.

[3] J. Banaś, J.Caballero, J.Rocha and K.Sadarangani, "Monotonic Solutions of a Class of Quadratic Integral Equations of Volterra Type". Computers and Mathematics with Applications, 49(2005), 943-952.

[4] J. Banaś, J. Rocha Martin and K. Sadarangani, "On the solution of a quadratic integral equation of Hammerstein type". Mathematical and Computer Modelling, 43 (2006), 97-104.

[5] J. Banaś and B. Rzepka, "Monotonic solution of a quadratic integral equations of fractional order". J. Math. Anal. Appl., 332(2007), 1370-11378.

[6] A.M.A EL-Sayed, M.M. Saleh and E.A.A. Ziada, "Numerical Analytic Solution for Nonlinear Quadratic Integral Equations". Math. Sci. Res. J., 12(8) (2008), 183-191.

[7] A.M.A EL-Sayed and H.H.G. Hashem, "Carathéodory type theorem for nonlinear quadratic integral equation". Math. Sci. Res. J., 12(4) (2008), 71-95.

[8] A.M.A EL-Sayed and H.H.G. Hashem, "Integrable and continuous solution of nonlinear quadratic integral equation". Electronic Journal of Qualitative Theory of Differential Equations, 25(2008), 1-10.

[9] A.M.A EL-Sayed and H.H.G. Hashem, "Monotonic positive solution of nonlinear quadratic integral equation Hammerstein and Urysohn functional integral equation". Commentationes Mathematicae, 48(2) (2008), 199-207.

[10] A.M.A EL-Sayed and H.H.G. Hashem, "Solvability of nonlinear Hammerstein quad- ratic integral equations". J. Nonlinear Sci. Appl., 2(3) (2009), 152-160.

[11] STEVEN R. LAY, "Convex Set and Their Applications". University Cleveland. New York. 2007.

[12] R. F. Curtain and A. J. Pritchard, "Functional Analysis in Modern Applied Mathematics", Academic press, 1977.

[13] A. N. Kolmogorov and S. V. fomin, "Introduction real Analysis", Dover Publ. Inc. 1975.

[14] V. Lakshmikantham and S. Leela, "Differential and integral inequalities", vol. 1, New York London, 1969.

[15] M. R. Reo, "Ordinary Differential Equations", East-West Press, 1980.