A NEW SUB CLASS OF UNIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH A MULTIPLIER LINEAR OPERATOR

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ABSTRACT: *This paper deals with a new class* $T_{k,\mu}^{m}(\alpha, A, B)$ *which is a subclass of uniformly starlikefunctions involving a multiplier linear operator* $\Im_{k,\mu}^{m}$. *Coefficients inequality, Distortion theorem, Extreme points, Radius of starlikeness and radius of convexity for functions belonging to this class are obtained.*

Key words and phrases: Univalent, starlike, subordination, convex, analytic, linear operator.
2010 Mathematics Subject Classification: 30C45.

1. INTRODUCTION

Let S denote the class of functions of the form

$$(1.1)\quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic and univalent in the unit open disk Δ={z: |z|<1}.
Silverman[9] had introduced and studied a subclass T of S consisting of functions of the form

$$(1.2)\quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, \forall n \geq 2)$$

Let f and g be analytic in Δ.Then g is said to be subordinate to f, written as $g \prec f$ or $g(z) \prec f(z)$, if there exists a Schwartz function ω, which is analytic in Δ with ω(0)=0 and $|\omega(z)| < 1 (z \in \Delta)$ such that $g(z) = f(\omega(z)) (z \in \Delta)$. In particular, if the function f is univalent in Δ, we have the following equivalence([3],[7]) $g(z) \prec f(z) (z \in \Delta) \Leftrightarrow g(0) = f(0) \text{ and } g(\Delta) \subseteq f(\Delta)$.

Sharma and Raina ([4],[5],[6]) have introduced a multiplier linear operator $\Im_{k,\mu}^{m}$ for $m \in Z, \mu > -1, k > 0$ by

$$(1.3)\quad \begin{cases} \Im_{k,\mu}^{m} f(z) = f(z), & m = 0, \\ \Im_{k,\mu}^{m} f(z) = \dfrac{\mu+1}{k} z^{1-\frac{\mu+1}{k}} \displaystyle\int_0^z t^{\frac{\mu+1}{k}-2} \Im_{k,\mu}^{m+1} f(t)dt, & m \in Z^- = \{-1,-2,........\} \\ \Im_{k,\mu}^{m} f(z) = \dfrac{k}{\mu+1} z^{2-\frac{\mu+1}{k}} \dfrac{d}{dt}\left(z^{\frac{\mu+1}{k}-1} \Im_{k,\mu}^{m-1} f(z)\right), & m \in Z^+ = \{1,2,..........] \end{cases}$$

The series representation of $\Im_{k,\mu}^{m} f(z)$ for f(z) of the form (1.1) is given by

$$(1.4)\quad \Im_{k,\mu}^{m} f(z) = z + \sum_{n=2}^{\infty} \left(1 + \frac{k(n-1)}{\mu+1}\right)^m a_n z^n,$$

Also

$(i)\ \Im_{k,0}^{m} = D_k^m, m \geq 0$ [1]

$(ii)\ \Im_{1,0}^{m} = D^m, m \geq 0$ [8]

$(iii)\ \Im_{1,1}^{m} = D^{m}, m \geq 0$ [10]

$(iv)\ \Im_{1,\mu}^{m} = I_{\mu}^{m}, m \geq 0$ [2]

Involving the operator $\Im_{k,\mu}^{m}$, we define a class $T_{k,\mu}^{m}(\alpha, A, B)$ as follows:

Definition 1.1: For fixed A, B, $-1 \leq B < A \leq 1, 0 < A \leq 1, 0 < \alpha \leq 1, z \in \Delta$, a function $f \in T$ is said to be in class $T_{k,\mu}^{m}(\alpha, A, B)$ if

$$(1.6)\quad \frac{z\left(\Im_{k,\mu}^{m} f(z)\right)'}{\Im_{k,\mu}^{m} f(z)} \prec \frac{1+[B+\alpha\{(1-\alpha)+(A-B)]z}{1+Bz}, z \in \Delta.$$

From the definition, it follows that $f \in T_{k,\mu}^{m}(\alpha, A, B)$ if there exists a function w(z) analytic in Δ and satisfies w(0)=0 and $|w(z)| < 1$ for $z \in \Delta$, such that

$$\frac{z\left(\Im_{k,\mu}^{m} f(z)\right)'}{\Im_{k,\mu}^{m} f(z)} = \frac{1+[B+\alpha\{(1-\alpha)+(A-B)]w(z)}{1+Bw(z)}, z \in \Delta.$$

or equivalently

$$(1.7)\quad \left| \frac{\dfrac{z\left(\Im_{k,\mu}^{m} f(z)\right)'}{\Im_{k,\mu}^{m} f(z)} - 1}{B+\alpha\{(1-\alpha)+(A-B) - B\dfrac{z\left(\Im_{k,\mu}^{m} f(z)\right)'}{\Im_{k,\mu}^{m} f(z)}} \right| < 1, z \in \Delta.$$

The main object of this paper is to obtain necessary and sufficient conditions for the functions $f(z) \in T_{k,\mu}^{m}(\alpha, A, B)$. Furthermore we obtain extreme points, distortion bounds, Closure properties, radius of starlikeness and convexity for $f(z) \in T_{k,\mu}^{m}(\alpha, A, B)$.

2. COEFFICIENTS INEQUALITY

Theorem2.1: A necessary and sufficient condition for f(z) of the form (1.1) to be in the class $T_{k,\mu}^{m}(\alpha, A, B)$ is that

$$(2.1)\quad \sum_{n=2}^{\infty}[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)a_n \leq \alpha\{(1-\alpha)+(A-B)\}.$$

$$where \quad \theta_{k,\mu}^{m}(n) = \left(1+\frac{k(n-1)}{\mu+1}\right)^{m} \quad for\,(-1 \leq B < A \leq 1, 0 < A \leq 1, 0 < \alpha \leq 1).$$

Proof: Assume that the inequality (2.1) holds true and |z|=1.Then we have

$$\left|z\left(\Im_{k,\mu}^{m} f(z)\right)' - \Im_{k,\mu}^{m} f(z)\right| - \left|\Im_{k,\mu}^{m} f(z)[B+\alpha\{(1-\alpha)+(A-B)\}] - Bz\left(\Im_{k,\mu}^{m} f(z)\right)'\right|$$

$$= \left|\left(z - \sum_{n=2}^{\infty} n\theta_{k,\mu}^{m}(n)a_n z^n\right) - \left(z - \sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n z^n\right)\right| -$$

$$\left|\left(z - \sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n z^n\right)[B+\alpha\{(1-\alpha)+(A-B)\}] - B\left(z - \sum_{n=2}^{\infty} n\theta_{k,\mu}^{m}(n)a_n z^n\right)\right|$$

$$= \left| -\sum_{n=2}^{\infty}(n-1)\theta_{k,\mu}^{m}(n)a_n z^n \right| -$$

$$\left| [\alpha\{(1-\alpha)+(A-B)\}]z - [B+\alpha\{(1-\alpha)+(A-B)\}]\sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n z^n + B\sum_{n=2}^{\infty} n\theta_{k,\mu}^{m}(n)a_n z^n \right|$$

$$\leq \sum_{n=2}^{\infty}(n-1)\theta_{k,\mu}^{m}(n)a_n - \alpha\{(1-\alpha)+(A-B)\} + \sum_{n=2}^{\infty}[-B(n-1)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)a_n$$

$$\leq \sum_{n=2}^{\infty}[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)a_n - \alpha\{(1-\alpha)+(A-B)\} \leq 0.$$: Conversely, suppose that the function f(z) defined by(1.1) be in the class $T_{k,\mu}^{m}(\alpha, A, B)$.

Then from (2.1), we have

$$\left| \frac{\dfrac{z\left(\Im_{k,\mu}^{m} f(z)\right)'}{\Im_{k,\mu}^{m} f(z)} - 1}{B+\alpha\{(1-\alpha)+(A-B)-B\dfrac{z\left(\Im_{k,\mu}^{m} f(z)\right)'}{\Im_{k,\mu}^{m} f(z)}} \right|$$

$$= \left| \frac{z\left(\Im_{k,\mu}^{m} f(z)\right)' - \Im_{k,\mu}^{m} f(z)}{[B+\alpha\{(1+\alpha)+(A-B)\}]\Im_{k,\mu}^{m} f(z) - Bz\left(\Im_{k,\mu}^{m} f(z)\right)'} \right|$$

$$= \left| \frac{-\sum_{n=2}^{\infty}(n-1)\theta_{k,\mu}^{m}(n)a_n z^n}{[B+\alpha\{(1+\alpha)+(A-B)\}]\left(z-\sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n z^n\right) - B\left(z-\sum_{n=2}^{\infty} n\theta_{k,\mu}^{m}(n)a_n z^n\right)} \right| < 1.$$

Since $\operatorname{Re}(z) \leq |z|$ for all, we have

$$(2.2)\ \operatorname{Re}\left\{ \frac{\sum_{n=2}^{\infty}(n-1)\theta_{k,\mu}^{m}(n)a_n z^n}{\alpha\{(1+\alpha)+(A-B)\}z - \sum_{n=2}^{\infty}[-B(n-1)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)a_n z^n} \right\} < 1$$

We choose the value of z on the real axis so that $\dfrac{z\left(\Im_{k,\mu}^{m} f(z)\right)'}{\Im_{k,\mu}^{m} f(z)}$ is real. Upon clearingthe denominator of (2. 2) and letting $z \to 1^-$, we can write (2.2) as $\sum_{n=2}^{\infty}[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)a_n \leq \alpha\{(1-\alpha)+(A-B)\}.$.

Thus (2.1) proves the theorem.

The result is sharp. The extremal function being

$$(2.3)\quad f(z) = z - \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)} z^n, n \geq 2.$$

Corollary 2.2: Let the function f(z) defined by(1.1) be in the class $T_{k,\mu}^{m}(\alpha, A, B)$.Then

(2.4) $$a_n \leq \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)}, n \geq 2.$$

3. DISTORTION THEOREMS

Theorem3.1: If f (z) $\in T_{k,\mu}^{m}(\alpha, A, B)$, then for $z \in \Delta$

(3.1) $$\left\| f(z)\right| - |z|\right| \leq \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\left(1+\frac{k}{\mu+1}\right)^m}|z|^2, n \in N$$

and

(3.2) $$\left\|\Im_{k,\mu}^{m} f(z)\right| - |z|\right| \leq \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]}|z|^2, n \in N.$$

Proof: In view of inequality (2.1), it follows that

$$\sum_{n=2}^{\infty}[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)a_n \leq \alpha\{(1-\alpha)+(A-B)\}.$$

(3.3) or $$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\left(1+\frac{k}{\mu+1}\right)^m}, n \geq 2.$$

Therefore

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$

or

(3.4) $$|f(z)| \geq |z| - \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\left(1+\frac{k}{\mu+1}\right)^m}|z|^2$$

and

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n$$

or

(3.5) $$|f(z)| \leq |z| + \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\left(1+\frac{k}{\mu+1}\right)^m}|z|^2$$

From (3.4) and (3.5) inequality (3.1) follows.
Further for f (z) $\in T_{k,\mu}^{m}(\alpha, A, B)$, the inequality (2.1) gives

$$[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n \leq \alpha\{(1-\alpha)+(A-B)\}.$$

(3.6) or $\sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n \le \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]}, n \ge 2.$

Thus,

$$\left|\Im_{k,\mu}^{m} f(z)\right| \ge |z| - \sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n|z|^n \ge |z| - |z|^2\sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n$$

or

(3.7) $$\left|\Im_{k,\mu}^{m} f(z)\right| \ge |z| - \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]}|z|^2$$

And $$\left|\Im_{k,\mu}^{m} f(z)\right| \le |z| + \sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n|z|^n \le |z| + |z|^2\sum_{n=2}^{\infty}\theta_{k,\mu}^{m}(n)a_n$$

or

(3.8) $$\left|\Im_{k,\mu}^{m} f(z)\right| \le |z| + \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(1-B)+\alpha\{(1-\alpha)+(A-B)\}]}|z|^2$$

On using (3.7) and (3.8) inequality (3.2) follows.

4. EXTREME POINTS

Theorem 4.1: Let

(4.1) $f_1(z) = z$ and $f_n(z) = z - \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)}z^n$

for $n \ge 2$,then f (z) $\in T_{k,\mu}^{m}(\alpha, A, B)$, if and only if it can be expressed in the form

(4.2) $f(z) = \sum_{n=1}^{\infty} d_n f_n(z)$, where $d_n \ge 0$ and $\sum_{n=1}^{\infty} d_n = 1$.

In particular the extreme points of $T_{k,\mu}^{m}(\alpha, A, B)$ are the functions given by (4.1).

Proof: Let f(z) be expressed in the form (4.1),then

$$f(z) = \sum_{n=1}^{\infty} d_n f_n(z) = z - \sum_{n=2}^{\infty}\frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)}z^n$$

$$= z - \sum_{n=2}^{\infty} d_n b_n z^n$$

Where $b_n = \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)}$

Now, since

$$\sum_{n=2}^{\infty}[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)d_n b_n = \sum_{n=2}^{\infty}\alpha\{(1-\alpha)+(A-B)\}d_n$$

$$= \alpha\{(1-\alpha)+(A-B)\}(1-d_1) \le \alpha\{(1-\alpha)+(A-B)\}.$$

Therefore, f (z)$\in T_{k,\mu}^{m}(\alpha, A, B)$.

Conversely, let f (z) $\in T_{k,\mu}^{m}(\alpha, A, B)$, then (2.1) yields

$$a_n \le \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^{m}(n)}z^n \text{ for } n \ge 2.$$

Setting $d_n = \frac{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^m(n)}{\alpha\{(1-\alpha)+(A-B)\}} a_n$ for $n \geq 2$

And $d_1 = 1 - \sum_{n=2}^{\infty} d_n$.

Then $f(z) = z - \sum_{n=2}^{\infty} \frac{\alpha\{(1-\alpha)+(A-B)\}}{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^m(n)} d_n z^n$

$$= z - \sum_{n=2}^{\infty} d_n\{z - f_n(z)\}$$

$$= z(1-\sum_{n=2}^{\infty} d_n) + \sum_{n=2}^{\infty} d_n f_n(z) = \sum_{n=1}^{\infty} d_n f_n(z).$$

This completes the proof.

5. RADIUS OF STARLIKENESS

THEOREM 5.1: Let f (z) ∈ $T_{k,\mu}^m(\alpha, A, B)$, then f(z) is starlike in $|z| < r(\alpha, A, B)$, *where*

(5.1) $r = \inf\left[\frac{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^m(n)}{n\alpha\{(1-\alpha)+(A-B)\}}\right]^{\frac{1}{n-1}}, n \geq 2.$

Proof: It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

i.e. $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}} < 1$

(5.2) or $\sum_{n=2}^{\infty} na_n|z|^{n-1} < 1.$

It is easily to see that (5.1) holds if

$|z|^{n-1} < \left[\frac{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^m(n)}{n\alpha\{(1-\alpha)+(A-B)\}}\right].$

This completes the proof.

6. RADIUS OF CONVEXITY

THEOREM 6.1: Let f (z) ∈ $T_{k,\mu}^m(\alpha, A, B)$, then f(z) is convex in $|z| < r(\alpha, A, B)$, *where*

(6.1) $r = \inf\left[\frac{[(n-1)(1-B)+\alpha\{(1-\alpha)+(A-B)\}]\theta_{k,\mu}^m(n)}{n^2\alpha\{(1-\alpha)+(A-B)\}}\right]^{\frac{1}{n-1}}, n \geq 2.$

Proof: Upon noting the fact that f(z) is convex if and only if $zf'(z)$ is starlike, the Theorem(6.1) follows.

REFERANCES

[1]. Al-Oboudi, F.M., "On univalent functions defined by a generalized Salageon operator". Indian J. Math, Math. Sci. 25-28(2004), 1429-1436.
[2]. Cho, N.E. and Kim, T.H., "Multiplier transformations and strongly close-to-convex functions", Bull. Korean Math. Soc. 40(2003), 399-410.
[3].Owa,S. and Srivastava,H.M., "Univalent and starlike generalized hypergeometric functions". Canad.J.Math.,39(5)(1987),1057-1077.
[4].Raina, R.K. and Sharma, P., "Subordination properties of univalent functions involving a new class of operators",Electr. J. Math. Anal. App. 2(1), (2014),37-52.
[5].Raina, R.K. and Sharma, P., "Subordination preserving properties associated with a class of operators", Le Mathematiche 68(1),(2013),217-228.
[6].Raina, R.K., Sharma, P. and Sokol,J. "Certain subordination results involving a class of operators". Analele Univ. Oradea Fasc. Matematica 21(2),(2014),89-99.
[7]. RUSCHEWEYH S., "Neighborhoods of Univalent Functions", Proc. Amer. Math., Soc., 81(1981), 521-527.
[8]. Salageon, G.S., "Subclasses of univalent function in complex analysis" v Romanian-Finnish seminar, Part- I (Bucharest, 1981), Lecture notes in Mathematics 1013, Springer Verlag, Berlin and New York (1983).
[9] Silverman, H., "Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51, (1975), 109-116.
[10]. Uralegaddi, B.A. and Somantha, C., " Certain classes of univalent functions in H.M. Srivastava, S. Owa (Eds.) Current topics in analytic function theory, World Scientific Publishing company, Singapore, (1992), 371-374.