Analogy between Additive Combinations, Finite incidence Geometry & Graph Theory

K SrnivasaRao[1], Dr k.Rameshbabu[2]

[1]Associate Professor, Mathematics, H&S.CIET, JNTUK, Guntur, AP, India.
[2]Associate. Professor, Comm. stream, ECE Department, JIT, JU, East.Afrika

---

Abstract: *The topic is the study of the Tur´an number for C4. Fu¨redi showed that C4-free graphs with ex(n,C4) edges are intimately related to polarity graphs of projective planes. Then prove a general theorem about dense subgraphs in a wide class of polarity graphs, and as a result give the best-known lower bounds for ex(n,C4) for many values of n.and also study the chromatic and independence numbers of polarity graphs, with special emphasis on the graph ERq. Next study is Sidon sets on graphs by considering what sets of integers may look like when certain pairs of them are restricted from having the same product. Other generalizations of Sidon sets are considered as well. Then use C4-free graphs to prove theorems related to solvability of equations. Given an algebraic structure R and a subset A ⊆ R, define the sum set and product set of A to be A+A = {a+b : a,b ∈ A} and A·A = {a·b : a,b ∈ A} respectively. Showing under what conditions at least one of |A + A| or |A · A| is large has a long history of study that continues to the present day. Using spectral properties of the bipartite incidence graph of a projective plane, we deduce that nontrivial sum-product estimates hold in the setting where R is a finite quasifield. Several related results are obtained.*

Keywords: **Tur´an, C4-free graphs, chromatic, Sidon sets, general theorem, projective plane etc**

I. INTRODUCTION

Let us know clearly about Additive Combinations, Finite incidence Geometry & Graph Theory in detail. Additive combinatoricsis a compelling and fast growing area of research in mathematical sciences, and the goal of this paper is to survey some of the recent developments and notable accomplishments of the field, focusing on both pure results and applications with a view towards computer science and cryptography. One might say that additive combinatorics studies combinatorial properties of algebraic objects, for example, Abelian groups, rings, or fields, and in fact, focuses on the interplay between combinatorics, number theory, harmonic analysis, ergodic theory, and some other branches. Green describes additive combinatorics as the following: “additive combinatorics is the study of approximate mathematical structures such as approximate groups, rings, fields, polynomials and homomorphisms”.Approximate groups can be viewed as finite subsets of a group with the property that they are almost closed under multiplication. Approximate groups and their applications (for example, to expand er graphs, group theory, Probability, model theory, and so on) form a very active and promising area of research in additive combinatorics. this paper describes additive combinatorics as the following: “additive combinatorics focuses on three classes of theorems: decomposition theorems, approximate structural theorems, and transference principles ”.Techniques and approaches applied in additive combinatorics are often extremely sophisticated, and may have roots in several unexpected fields of mathematical sciences, Additive combinatorics has recently found a great deal of remarkable applications to computer science and cryptography.
Methods from additive combinatorics provide strong techniques for studying the so-called
threshold phenomena, which is itself of significant importance in combinatorics, computer science, discrete probability, statistical physics, and economics. Additive combinatorics has seen very fast advancements in the wake of extremely deep work on Szemer´edi’s theorem, the proof of the existence of long APs in the primes by Greenand Tao, and generalizations and applications of the sum-product problem, and continues to see significant progress .

A finite geometry is any geometric system that has only a finite number of points. The familiar Euclidean geometry is not finite, because a Euclidean line contains infinitely many points. A geometry based on the graphics displayed on a computer screen, where the pixels are considered to be the points, would be a finite geometry. While there are many systems that could be called finite geometries, attention is mostly paid to the finite projective and affine spaces because of their regularity and simplicity. Other significant types of finite geometry are finite Möbius or inversive planes and Laguerre planes, which are examples of a general type called Benz planes, and their higher-dimensional analogs such as higher finite geometries. Finite geometries may be constructed via linear algebra, starting from vector spaces over a finite field; the affine and projective planes so constructed are called Galois geometries. Finite geometries can also be defined purely axiomatically. Most common finite geometries are Galois geometries, since any finite projective space of dimension three or greater is isomorphic to a projective space over a finite field (that is, the projectivization of a vector space over a finite field). However, dimension two has affine and projective planes that are not isomorphic to Galois geometries, namely the non-Desarguesian planes. Similar results hold for other kinds of finite geometries.

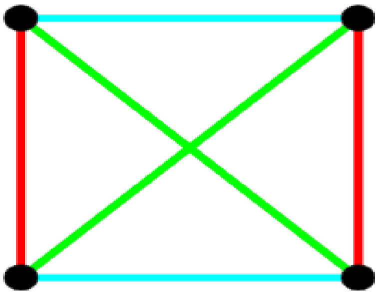

Fig 1. Finite affine plane of order 2, containing 4 points and 6 lines. Lines of the same color are "parallel".

I.1Classification of finite projective spaces by geometric dimension

i))Dimension 0 (no lines): The space is a single point and is so degenerate that it is usually ignored.

ii)Dimension 1 (exactly one line): All points lie on the unique line, called a *projective line*.

iii) Dimension 2: There are at least 2 lines, and any two lines meet. A projective space for $n = 2$ is a projective plane. These are much harder to classify, as not all of them are isomorphic with a PG(d, q). The Desarguesian planes (those that are isomorphic with a PG(2, q)) satisfy Desargues's theorem and are projective planes over finite fields, but there are many non-Desarguesian planes.

iv)Dimension at least 3: Two non-intersecting lines exist. The Veblen–Young theorem states in the finite case that every projective space of geometric dimension $n \geq 3$ is isomorphic with a PG(n, q), the n-dimensional projective space over some finite field GF(q).

v)In mathematics, incidence geometry is the study of incidence structures. A geometric structure such as the Euclidean plane is a complicated object that involves concepts such as length, angles, continuity, betweenness, and incidence. An *incidence structure* is what is obtained when all other concepts are removed and all that remains is the data about which points lie on which lines. Even with this severe limitation, theorems can be proved and interesting facts emerge concerning this structure. Such fundamental results remain valid when additional concepts are added to form a richer geometry. It sometimes happens that authors blur the distinction between a study and the objects of that study, so it is not surprising to find that some authors refer to incidence structures as incidence geometries.[1]

vi)Incidence structures arise naturally and have been studied in various areas of mathematics. Consequently there are different terminologies to describe these objects. In graph theory they are called hypergraphs, and in combinatorial design theory they are called block designs. Besides the difference in terminology, each area approaches the subject differently and is interested in questions about these objects relevant to that discipline. Using geometric language, as is done in incidence geometry, shapes the topics and examples that are normally presented. It is, however, possible to translate the results from one discipline into the terminology of another, but this often leads to awkward and convoluted statements that do not appear to be natural outgrowths of the topics. In the examples selected for this article we use only those with a natural geometric flavour.

vi)A special case that has generated much interest deals with finite sets of points in the Euclidean plane and what can be said about the number and types of (straight) lines they determine. Some results of this situation can extend to more general settings since only incidence properties are considered.

vii)In mathematics graph theory is the study of *graphs*, which are mathematical structures used to model pair wise relations between objects. A graph in this context is made up of *vertices*, *nodes*, or *points* which are connected by *edges*, *arcs*, or *lines*. A graph may be *undirected*, meaning that there is no distinction between the two vertices associated with each edge, or its edges may be *directed* from one vertex to another; see Graph (discrete mathematics) for more detailed definitions and for other variations in the types of graph that are commonly considered. Graphs are one of the prime objects of study in discrete mathematics.

viii)Refer to the glossary of graph theory for basic definitions in graph theory.

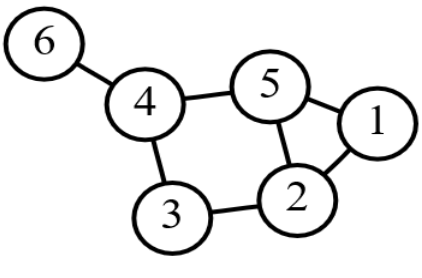

Fig 2 A drawing of Graph.

2 THEORMS STUDY

Theorem.1 If q is a prime power, then ex(q2 −q−2,C4) ≥ 1 2q3 −q2 +q−Oq1/2. From here, we study properties of polarity graphs without Tura´n numbers in mind. One polarity graph of particular interest is the graph ERq. If q is a prime power, the vertices of ERq are the one-dimensional subspaces of a three dimensional vector space over Fq, and two distinct subspaces are adjacent if they are orthogonal to each other. The graph ERq has been studied in a variety of settings,, we study the chromatic number of ERq. In particular, we prove the following theorem, which is best possible up to the constant 2.

Theorem 2. Let Π be a projective plane of order q that contains an oval and has a polarity π. If m ∈ {1,2,...,q + 1}, then the polarity graph Gπ contains a sub graph on at most m +m/2 vertices that has at least 2(m/ 2)+ (m^4 /8q) −O((m4 /$q^{3/2}$) + m) edges.

Theorem 3. (Hoffman [50]). Let G be a d-regular graph on n vertices and λ_n be the smallest eigenvalue of its adjacency matrix. Then α(G) ≤ n(−λn / d−λn) . As the graph ERq with loops on the absolute points is regular, Hoffman's theorem may be applied to obtain α(ERq) ≤ q3/2 + q1/2 + 1. Therefore, the order of magnitude of α(ERq) is $q^{3/2}$. Godsil and Newman refined the upper bound obtained from Hoffman's bound. Their result was then improved using the Lova´sz theta function. When q is even, Hobart and Williford used coherent configurations to provide upper bounds for the independence number of general orthogonal polarity graphs. When q is an even square, the known upper bound and lower bound for α(ERq) differ by at most 1. In the case when p is odd or when p = 2 and n is odd, it is still an open problem to determine an asymptotic formula for α(ERq).

Since the independence number has been well-studied and its order of magnitude is known, it is natural to investigate the chromatic number of ERq which is closely related to α(ERq). Let q be any prime power. Then ERq has q2 + q + 1 vertices and α(ERq) = Θ(q3/2), and so a lower bound for χ(ERq) is q2+q+1 α(ERq) ≥ q1/2. One may ask whether this lower bound actually gives the right order of magnitude of χ (ERq). We confirm this for q being an even power of an odd prime.

Theorem 4.. If q is a power of an odd prime and f(X) ∈Fq^2[X] is a planar polynomial all of whose coefficients belong to the subfield Fq, then α (Gf) ≥q^2(q−1) Even though we have the restriction that the coefficients of f belong to Fq, many of the known examples of planar functions have this property. Most discussed including those that give rise to the famous Coulter-Matthews plane, satisfy our requirement. It is still an open problem to determine an asymptotic formula for the independence number of ERp for odd prime p. However, Theorem 4. it would be quite surprising to find an orthogonal polarity graph of a projective plane of order q whose independence number is o($q^{3/2}$). We believe that the lower bound Ω($q^{3/2}$) is a property shared by all polarity graphs, including polarity graphs that come from polarities which are not orthogonal.

Theorem 5. There exist constants a,b > 0 such that (i) P(n,d) ∼ n if d ≤ $n^{1/2}$(log n)−a and (ii) P(n,d) ∼ n log n if d ≥ n1/2(log n)b. An old result of Erdo˝s implies P(Kn) ∼ n log n, whereas Theorem 5.shows that P(G) ∼ n log n for graphs which are much sparser than Kn. The labeling of the vertices of any n-vertex graph G with the first n prime numbers is always a product-injective labeling from [N] where N ∼ nlog n, via the Prime Number Theorem. Theorem 5. then is an analogous result for products that Bolloba´s and Pikhurko proved for sums and differences, where a change in behavior was also observed around d = n1/2. Theorem 5. will be proved with a > log2 and b > 5.5; while our method allows these values to be slightly improved, new ideas would be needed to determine P(n,d) for all the intermediate values of d. In fact, the proof of Theorem 5.(ii) establishes the much stronger result that if G is the random graph on n vertices with edge-probability d/n and d ≥ n1/2(logn)b, then P(G) ∼ nlogn almost surely as n →∞. We also remark that Theorem 5. determines the maximum value of P(G) over n-vertex graphs with m edges for almost all possible values of m. It is a natural question to ask about other functions besides sums and products. Let H be a k-uniform hypergraph and denote by Kk n the complete k-uniform hypergraph on n vertices. If φ is a general symmetric function of k variables, then Sφ(H) is the minimum integer N such that there exists an injection c :V (H) → [N] such that whenever {v1,v2,...,vk},{w1,w2,...,wk} ∈ E(H) are distinct hyperedges, φ(c(v1),c(v2),...,c(vk)) 6= φ(c(w1),c(w2),...,c(wk)). The question of determining S(G) and P(G) then is the case when k = 2 and φ(x,y) = x + y or φ(x,y) = x·y respectively. For general H and φ, this quantity depends on number theoretic questions involving the number of representations of integers as evaluations of φ. For instance, if k = 2 and φ(x,y) = (xy)2 + x + y and φ(x,y) = φ(u,v), then it is not hard to show {x,y} = {u,v} and therefore Sφ(G) = n for every n-vertex graph G. Define Rφ(N) = X x,y∈[N]k x6=y 1φ(x)=φ(y). This is the number of ways of writing integers in two ways as evaluations of φ on [N]k. We prove the following general theorem:

Theorem 6. Let q be a power of an odd prime. If A ⊂Fq, |A+A| = m and |A·A| = n,

then $|A|^2 \leq mn\,|A|/q + q^{1/2}\sqrt{mn}$.

Theorem 7. Let Π be a finite projective plane of even order which admits an orthogonal polarity. Then Π contains a Fano sub plane.

Ganley showed that a finite semifield plane admits an orthogonal polarity if and only if it can be coordinatized by a commutative semifield. A result of Kantor implies that the number of no isomorphic planes of order n a power of 2 that can be coordinatized by a commutative semifield is not bounded above by any polynomial in n. Thus, Theorem 7. Applies to many projective planes

3 CONCLUSION

First, we note that the proof of Theorem (Let Π be a finite projective plane of even order which admits an orthogonal polarity. Then Π contains a Fano sub plane) actually implies that there are $\Omega(n3)$ copies of PG(2,2) in any plane satisfying the hypotheses, and echoing Petrak , perhaps one could find sub planes of order 4 for n large enough. We also note that it is crucial in both proofs that the absolute points form a line. When n is odd, the proof fails (as it must, since the proofs do not detect if Π is Desargesian or not). Finally, Bill Kantor communicated to the author that Theorem can be proved in an even shorter way by using the language of finite incidence geometry and a theorem of Ostrom (Theorem There exist constants a,b > 0 such that (i) $P(n,d) \sim n$ if $d \leq n1/2(\log n)-a$ and (ii) $P(n,d) \sim n\log n$ if $d \geq n1/2(\log n)b$.). Using this theorem, it suffices to find a self-polar triangle in the projective plane (equivalent to finding the triangle vivjvk in our proof). An advantage of our proof is that we get an explicit lower bound on the number of Fano subplanes in the projective plane, whereas an advantage of Kantor's proof is that one does not even require the projective plane to be finite.

Theorem 7.. Let Π be a finite projective plane of even order which admits an orthogonal polarity. Then Π contains a Fano sub plane.

Ganley showed that a finite semifield plane admits an orthogonal polarity if and only if it can be coordinatized by a commutative semifield. A result of Kantor implies that the number of no isomorphic planes of order n a power of 2 that can be coordinatized by a commutative semifield is not bounded above by any polynomial in n. Thus, Theorem 7. Applies to many projective planes.

There are two special circumstances in which one can improve Theorem 2. Each indicates the difficulty of finding exact values for the parameter ex(n,C4).

The first situation is when q is a square. In this case, Fq contains the subfield F√q and this subfield may be used to find small graphs that contain many edges. For instance ERq contains a subgraph F that is isomorphic to ER√q. One can choose $m = \sqrt{q} + 1$ and let S be the set of absolute points in F. These m vertices will also be absolute points in ERq and thus are contained in an oval (the absolute points of an orthogonal polarity of PG(2,q) form an oval when q is odd). If we then consider them 2vertices in YS, these will be the vertices in F that are adjacent to the absolute points of F. The set YS induces a 1 2(√q −1)-regular graph in F . The set X = S ∪YS will span roughly q3/2 8 edges which is much larger than the linear in q lower bound provided by Theorem 2.1.1 when $m = \sqrt{q} + 1$.

The second situation is when q is a power of 2 and q −1 is prime. Assume that this is the case and consider ERq−1. Let F be a sub graph of ERq−1 obtained by deleting three vertices of degree q −1. The number of vertices of F is (q−1)2 +(q−1)+1−3 = q2−q−2, and the number of edges of F is at least 1 2(q−1)q2−3(q−1) = 1 2q3− 1 2q2−3q +3. This is better than the result of Corollary 2.1.2 by a factor of about 1 2q2. A prime of the form 2m−1 with m ∈ N is known as a Mersenne Prime. It has been conjectured that there are infinitely many such primes, but this is a difficult open problem.

In Sidon sets are used to construct C4-free graphs. For a prime power q, these graphs have q2−1 vertices, and 1 2q3−q+ 1 2 edges when q is odd, and 1 2q3−q edges when q is even. These graphs have a degree sequence similar to the degree sequence of an orthgonal polarity graph and it seemed possible that these graphs could be external. However, Theorem 2. can be applied to show ex(q2 −1,C4) ≥ 1 2 q3 −O(√q), which shows that the graphs constructed are not extremal.

First, we note that the proof of Theorem 7. actually implies that there are $\Omega(n3)$ copies of PG(2,2) in any plane satisfying the hypotheses, and echoing Petrak , perhaps one could find sub planes of order 4 for n large enough. We also note that it is crucial in both proofs that the absolute points form a line. When n is odd, the proof fails (as it must, since the proofs do not detect if Π is Desargesian or not). Finally, Bill Kantor communicated to the author that Theorem 7. can be proved in an even shorter way by using the language of finite incidence geometry and a theorem of Ostrom (Theorem 5). Using this theorem, it suffices to find a self-polar triangle in the projective plane (equivalent to finding the triangle vivjvk in our proof). An advantage of our proof is that we get an explicit lower bound on the number of Fano sub planes in the projective plane, whereas an advantage of Kantor's proof is that one does not even require the projective plane to be finite.

4 BIBLIOGRAPHY

[1] Marien Abreu, Camino Balbuena, and Domenico Labbate. Adjacency matrices of polarity graphs and of other c 4-free graphs of large size. Designs, Codes and Cryptography, 55(2-3):221–233, 2010.

[2] Peter Allen, Peter Keevash, Benny Sudakov, and Jacques Verstra¨ete. Tura´n numbers of bipartite graphs plus an odd cycle. Journal of Combinatorial Theory, Series B, 106:134–162, 2014.

[3] Noga Alon and Fan RK Chung. Explicit construction of linear sized tolerant networks. Annals of Discrete Mathematics, 38:15–19, 1988.

[4] Noga Alon and Paul Erd˝os. An application of graph theory to additive number theory. European Journal of Combinatorics, 6(3):201–203, 1985.

[5] Noga Alon, Michael Krivelevich, and Benny Sudakov. Coloring graphs with sparse neighborhoods. Journal of Combinatorial Theory, Series B, 77(1):73– 82, 1999.

[6] Noga Alon and Vojtˇech Ro¨dl. Sharp bounds for some multicolor ramsey numbers. Combinatorica, 25(2):125–141, 2005.

[7] Noga Alon and Joel H Spencer. The probabilistic method. John Wiley & Sons, 2000. [8] Martin Bachraty` and Jozef ˇSira´nˇ. Polarity graphs revisited. Ars Mathematica Contemporanea, 8(1), 2014.

[9] Reinhold Baer. Projectivities with fixed points on every line of the plane. Bulletin of the American Mathematical Society, 52(4):273–286, 1946.

[10] B´ela Bollob´as and Oleg Pikhurko. Integer sets with prescribed pairwise differences being distinct. European Journal of Combinatorics, 26(5):607–616, 2005.

[11] Anthony Bonato and Andrea Burgess. Cops and robbers on graphs based on designs. Journal of Combinatorial Designs, 21(9):404–418, 2013.

5. Authors Biography:

Mr K Srinivasarao[1] who is main author of this paper working as Associate.Professor,H&S HOD,CIET,Guntur AP.having almost 19 Years of teaching & Adminastrator.having Good command in teaching. he has Degrees B.sc(computer),M.sc(Mathematics) from Acharya Nagarjuna Univesity,parsuing PhD from annamalia.University.

Dr K. Rameshbabu[3] working as Associate. Professor, in Communication stream, ECE, JIT, Jimma, East Africa, He did B.E(ece), M.Tech, Ph.D having 20+ years of Experience in the field of Teaching completely. he is Guest professor for various colleges affiliated universities likeAnna.University,Shivajiuniiversity,JNTU,Pondicherry etc.he is member of IEEE,ISTE,VSI.IJCSIT, IRED he is reviewer for many journals & published 30+ papers in international journals and conferences & also conducted, attended many Faculty Development Programmers .his area of interest image processing,VLSI,digital electronics etc.