FOURIER SERIES INVOLVING H–FUNCTION OF TWO VARIABLES

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Abstract

The object of this paper is to establish some new Fourier series involving H–function of two variables.

1. Introduction:

Recently Mittal and Gupta [1, p. 117] has given the following notation of the H-function of two variables as:

$$H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)}$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, (i = 1, 2, 3; j = 2, 3). Also, all the A's, α's, B's, β's, γ's, δ's, E's, and F's are assumed to the positive quantities for standardization purpose.

The contour L_1 is in the ξ-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j\xi)$ (j = 1, ..., m_2) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ (j = 1, ..., n_2), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n_1) to the left of the contour.

The contour L_2 is in the η-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ (j = 1, ..., m_3) lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ (j = 1, ..., n_3), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n_1) to the left of the contour.

The function, defined by (1), is analytic function of x and y if

$R = \sum_{j=1}^{p_1}\alpha_j + \sum_{j=1}^{p_2}\gamma_j - \sum_{j=1}^{q_1}\beta_j - \sum_{j=1}^{q_2}\delta_j < 0,$

$R = \sum_{j=1}^{p_1}A_j + \sum_{j=1}^{p_3}F_j - \sum_{j=1}^{q_1}B_j - \sum_{j=1}^{q_3}F_j < 0,$

The H-function of two variables given by (1) is convergent if

$$U = -\sum_{j=n_1+1}^{p_1}\alpha_j - \sum_{j=1}^{q_1}\beta_j + \sum_{j=1}^{m_2}\delta_j - \sum_{j=m_2+1}^{q_2}\delta_j + \sum_{j=1}^{n_2}\gamma_j - \sum_{j=n_2+1}^{p_2}\gamma_j > 0, \qquad (2)$$

$$V = -\sum_{j=n_1+1}^{p_1}A_j - \sum_{j=1}^{q_1}B_j + \sum_{j=1}^{m_3}F_j - \sum_{j=m_3+1}^{q_3}F_j + \sum_{j=1}^{n_3}E_j - \sum_{j=n_3+1}^{p_3}E_j > 0, \qquad (3)$$

and $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$

2. Result Required:

The following results are required in our present investigation:

From Macrobert [2, 3]:

$$\frac{\sqrt{\pi}\Gamma(2-s)}{2\Gamma(\frac{3}{2}-s)}(\sin\theta)^{1-2s} = \sum_{r=0}^{\infty}\frac{(s)_r}{(2-s)_r}\sin(2r+1)\,\theta, \quad (4)$$

where $0 < \theta \leq \pi$, Re $s \leq \frac{1}{2}$.

$$\frac{\sqrt{\pi}\Gamma(1-s)}{\Gamma(\frac{1}{2}-s)}\left(\sin\frac{\theta}{2}\right)^{-2s} = 1 + 2\sum_{r=0}^{\infty}\frac{(s)_r}{(1-s)_r}\cos r\theta, \quad (5)$$

where $0 < \theta \leq \pi$.

3. Main Result:

In this paper we will establish the following Fourier series:

$$\sum_{r=0}^{\infty} H^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_1,q_1;p_2+2,q_2+2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(1-r,1),(c_j,\gamma_j)_{1,p_2},(2+r,1):(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(\frac{3}{2},1),(d_j,\delta_j)_{1,q_2},(1,1):(f_j,F_j)_{1,q_3}\end{matrix}\right]\sin(2r+1)\,\theta$$

$$= \frac{\sqrt{\pi}}{2}\sin\theta.\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix}x/\sin^2\theta\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right] \quad (6)$$

provided that $0 < \theta \leq \pi$, $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$, where U and V are given in (2) and (3) respectively.

$$H^{0,n_1;m_2+1,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1,1):(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(\frac{1}{2},1),(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$+2\sum_{r=0}^{\infty} H^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_1,q_1;p_2+2,q_2+2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(1-r,1),(c_j,\gamma_j)_{1,p_2},(1+r,1):(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(\frac{1}{2},1),(d_j,\delta_j)_{1,q_2},(1,1):(f_j,F_j)_{1,q_3}\end{matrix}\right]\cos r\theta$$

$$= \sqrt{\pi}H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix}x/\sin^2\frac{\theta}{2}\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right], \quad (7)$$

provided that $0 < \theta \leq \pi$, $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$, where U and V are given in (2) and (3) respectively.

Proof:

To prove (6), expressing the H-function on the left-hand side as Mellin-Barnes type integral (1), we have

$$\sum_{r=0}^{\infty}\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\frac{\Gamma(\frac{3}{2}-s)\Gamma(r+s)}{\Gamma(s)\Gamma(2+r-s)}x^{\xi}y^{\eta}\sin(2r+1)\,\theta\, d\xi d\eta$$

On changing the order of integration and summation which is easily seen to be justified, the above expression becomes

$$\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\frac{\Gamma(\frac{3}{2}-s)}{\Gamma(2-s)}\times$$
$$\times\left[\sum_{r=0}^{\infty}\frac{(s)_r}{(2-s)_r}\sin(2r+1)\,\theta\right]x^{\xi}y^{\eta}d\xi d\eta.$$

and on using the relation (4), it takes the form

$$\frac{\sqrt{\pi}}{2} sin\theta . \frac{(-1)}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1 (\xi,\eta)\theta_2(\xi)\theta_3(\eta)(x/\sin^2\theta)^{\xi} y^{\eta} d\xi d\eta.$$

which is just the expression on the right side of (6). (6) is the Fourier sine series for the H-function of two variables.

The Fourier cosine series (7) is proved in an analogous by using (5).

4. Special Cases:

On specializing the parameters in main results, we get following Fourier series in terms of H-function of one variable, which is a result given by Nigam [4, p. 53 (1.1) and (1.2)]:

$$\sum_{r=0}^{\infty} H_{p+2,q+2}^{m+1,n+1} \left[x\middle|_{\left(\frac{3}{2},1\right),(b_j,\beta_j)_{1,q},(1,1)}^{(1-r,1),(a_j,\alpha_j)_{1,p},(2+r,1)}\right] \sin(2r+1)\,\theta$$

$$= \frac{\sqrt{\pi}}{2} sin\theta . H_{p,q}^{m,n}\left[x/\sin^2\theta\middle|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] \quad (8)$$

provided that $0 < \theta \leq \pi$, $|argx| < ½\pi A$, where A is given by $\sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \equiv A > 0$.

$$H_{p+1,q+1}^{m+1,n} \left[x\middle|_{\left(\frac{1}{2},1\right),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(1,1)}\right] + 2\sum_{r=0}^{\infty} H_{p+2,q+2}^{m+1,n+1} \left[x\middle|_{\left(\frac{1}{2},1\right),(b_j,\beta_j)_{1,q},(1,1)}^{(1-r,1),(a_j,\alpha_j)_{1,p},(1+r,1)}\right]\cos r\theta$$

$$= \sqrt{\pi} H_{p,q}^{m,n} \left[x/\sin^2\frac{\theta}{2}\middle|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right], \quad (9)$$

provided that $0 < \theta \leq \pi$, $|argx| < ½\pi A$, where A is given by $\sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \equiv A > 0$.

References

1. Mittal, P. K. and Gupta Gupta, K. C.: An integral involving generalized function of two variables, Proc. Indian Acad. Sci., 75 A, p. 117-123.
2. Macrobert, T. M.: Fourier series for E-function, Math. Zeitsclie, 75, 79-82 (1961).
3. Macrobert, T. M.: Infinite series for E-function, Math. Zeitsclie, 71, 143-154 (1959).
4. Nigam, H. N.: Fourier series for Fox's H-Functions, İstanbul University Science Faculty The Journal of Mathematics, Physics and Astronomy Vol. 34 (1969), 53-58.