On the definition of fuzzy subgroup

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Abstract - *A fuzzy subset μ of a group G is called a fuzzy subgroup of G if (i) the fuzzy value μ(xy) of the product of any two elements x,y of G is not less than the minimum of the fuzzy values μ(x), μ(y), and (ii) the fuzzy value $\mu(x^{-1})$ of the inverse of an element x is equal to the fuzzy value μ(x) of the element. We prove that second condition can be derived from the first if and only if the group is torsion*

Key Words: Algebra, Fuzzy subgroup.

1. INTRODUCTION

The notion of fuzzy subsets was formulated by Zadeh [3] in 1965. Since then, fuzzy subsets have been applied to various branches of mathematics and computer science. Many other mathematicians introduced the concept of fuzzy in another important mathematical structures, like topological spaces, algebras, groups and graphs et cetera. In 1971, Rosenfeld [2] introduced the concept of fuzzy subgroups and since then, several authors have studied fuzzy subgroups. A comprehensive account of Fuzzy Group Theory and list of references can be found in the interesting book by Mordeson, Bhutani and Rosenfeld [1]. Numerous standard definitions, concepts, properties and results in group theory have counterparts in fuzzy group theory. But the fuzzy analogue of the following simple and elementary result in group theory is not available in fuzzy group theory literature:

Theorem A Let H be a non-empty subset of a group G. Then for all x, y ∈ H

$$xy \in H \text{ implies } x^{-1} \in H$$

if and only if *H* is torsion.

A subset *H* of a group *G* is called a subgroup of *G* if *H* itself is a group with respect to the operation defined in *G*. In order to establish that a non-empty subset *H* of an arbitrary group *G* is a subgroup of *G*, it is necessary and sufficient to verify the following two conditions:

(i) the product of any two elements of *H* is again in *H*, and

(ii) the inverse of each element of *H* is also again in *H*.

If *H* is finite or more generally torsion (A group *G* is called a torsion group if each element of *G* is of finite order), then the verification of (ii) is unnecessary because if an element *x* of order *n* is in *H*, then by (i) $x^{-1} = x^{n-1}$ is also in *H*. But, if *H* is not torsion, then both the conditions are necessary to verify. For example, the subset **N** of natural numbers fails to become a subgroup of the additive group **Z** of integers. The subset **N** satisfies (i) but not (ii). The purpose of this note is to prove the following fuzzy analogue of Theorem A:

Theorem B Let μ be any fuzzy subset of a group *G*. Then, for all $x, y \in G$,

$$\mu(xy) \geq \min.\{\mu(x), \mu(y)\} \text{ implies } \mu(x^{-1}) = \mu(x)$$

if and only if *G* is torsion.

2. PROOF OF THEOREM B

We first recall some basic definitions from fuzzy group theory for the sake of completeness.

Definition 2.1 A fuzzy subset μ of a set *S* is a function

$$\mu : S \to [0, 1].$$

Definition 2.2 A fuzzy subset μ of a group *G* is called a fuzzy subgroup of *G* if

(i) $\mu(xy) \geq \min.\{\mu(x), \mu(y)\}$, and

(ii) $\mu(x^{-1}) = \mu(x)$ for all $x, y \in G$.

It is easy to see that the definition 2.2 is equivalent to the following definition:

Definition 2.3 A fuzzy subset μ of a group *G* is called a fuzzy subgroup of *G* if

(i) $\mu(xy) \geq \min.\{\mu(x), \mu(y)\}$, and

(ii) $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$.

Definition 2.4 A group is called a torsion group (or periodic group) if each of its elements is of finite order.

Proof of Theorem B: First suppose that G is a torsion group and μ is a fuzzy subset of G such that

$$\mu(xy) = \min. \{\mu(x), \mu(y)\}$$

for all $x, y \in G$. It then follows that

$$\mu(x^n) \geq \mu(x) \text{ for all } n \geq 1.$$

Observe that if $x = e$, where e denotes the identity of G, then trivially

$$\mu(x^{-1}) \geq \mu(x).$$

Let $x \in G$ be of finite order $n \geq 2$. Then $x^{-1} = x^{n-1}$ and hence

$$\mu(x^{-1}) = \mu(x^{n-1}) \geq \mu(x).$$

Conversely, suppose that for any fuzzy subset μ of G and for all $x, y \in G$,

$$\mu(xy) \geq \min. \{\mu(x), \mu(y)\}$$

implies that

$$\mu(x^{-1}) \geq \mu(x).$$

We proceed to prove that G is a torsion group. On the contrary, suppose that G is not torsion. Then G has an element g of infinite order. Consider the fuzzy subset μ of G defined as follows:

$$\mu(e) = 1$$

$$\mu(g^m) = m/(m + 1) \text{ if } m \geq 1$$

$$\mu(h) = 0 \text{ if } h \in G - L,$$

where $L = \{g, g^2, g^3, \ldots\}$. Let $x, y \in G$. If one or both of x and y are e, then

$$\mu(y) = \min. \{\mu(x), \mu(y)\}.$$

Suppose therefore that neither of x and y is e. If both $x, y \in L$, then clearly

$$\mu(xy) \geq \min. \{\mu(x), \mu(y)\}.$$

If one or both of x and y are not in L, then

$$\min. \{\mu(x), \mu(y)\} = 0 \leq \mu(xy).$$

Therefore, in all possible cases,

$$\mu(xy) \geq \min. \{\mu(x), \mu(y)\}$$

for all $x, y \in G$. But if $m \geq 1$, then

$$\mu((g^m)^{-1}) = \mu(g^{-m}) = 0 < m/(m + 1) = \mu(g^m).$$

Thus μ is a fuzzy subset of G such that

$$\mu(xy) \geq \min. \{\mu(x), \mu(y)\}$$

for all $x, y \in G$, but there are some elements $x \in G$ for which

$$\mu(x^{-1}) < \mu(x).$$

This is a contradiction to assumption and hence G is torsion.

REFERENCES

[1] J. N. Mordeson, K. R. Bhutani and A. Rosenfeld, Fuzzy Group Theory, Springer-Verlag Berlin Heidelberg, 2005.

[2] A. Rosenfeld, "Fuzzy groups," J. Math. Anal. Appl., vol. 35 1971, pp. 512-517.

[3] L. A. Zadeh, "Fuzzy sets," Inform. and Control, vol. 8 1965, pp. 338-353.