ASYMPTOTIC PROPERTIES OF THE DISCRETE STABILITY TIME SERIES WITH MISSED OBSERVATIONS BETWEEN TWO-VECTOR VALUED STOCHASTIC PROCESS

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Abstract - In this paper, we defined the Expanded finite Fourier transform of the strictly stability \((r+s)\) vector valued time series where there are some randomly missed observations, asymptotic moments are derived and the application will be studied.

Key Words: Discrete time stability processes, Data tapers, Finite Fourier transform, Missing values, Complex Normal Distribution.

1. INTRODUCTION

Many authors, as e.g. Brillinger [1]; Dahlhaus[3]; Ghazal and Farag [4] studied "The estimation of the spectral density, autocovariance function and spectral measure of continuous time stationary processes"; E.A,El-Desokey[9] studied "Some properties of the discrete expanded finite Fourier transform with missed observations"; M.A.Ghazal, G.S. Mokaddis and A.El-Desokey[10],[11] are Studied "The Spectral Analysis of strictly stationary continuous time series" and "Asymptotic Properties of spectral Estimates of Second-Order with Missed Observations". The paper is organized as the following: Section1. Introduction, we develop asymptotic properties of estimates the desired \(\mu\), \(a(u)\) In Section 2, the Asymptotic properties of Expanded finite Fourier transform with missed observations was discussed in section 3, section 4 we will apply our theoretical study in two cases in climate and economy.

2. ASYMPTOTIC PROPERTIES OF ESTIMATES

THE DESIRED \(\mu\), \(a(u)\)

Consider an \((r+s)\) vector-valued stability series

\[ Z(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} , \quad (2.1) \]

\(t = 0, \pm 1, \pm 2, \ldots \) with \(X(t)\) \(r\) vector-valued and \(Y(t)\) \(s\) vector-valued.

We assume the series (2.1) is \((r+s)\) stability vector-valued series with components \(X(t), Y(t), j = 1, 2, \ldots, r, i = 1, 2, \ldots, s\) all of whose moments exist, we define the means as

\[ EX(t) = C_x , \quad EY(t) = C_y \]

The covariances

\[ E\left\{ X(t+u) - C_x \right\} \left\{ X(t) - C_x \right\}^T = C_{xx}(u), \]
\[ E\left\{ Y(t+u) - C_y \right\} \left\{ Y(t) - C_y \right\}^T = C_{yy}(u), \quad (2.3) \]

and the second-order spectral densities

\[ f_{xx}(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} C_{xx}(u) \exp(-i\lambda u) , \]
\[ f_{yy}(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} C_{yy}(u) \exp(-i\lambda u) , \quad (2.4) \]

\[ f_{xy}(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} C_{xy}(u) \exp(-i\lambda u) , \]

for \(-\infty < \lambda < \infty\).

In this section we consider the problem of determining an \(s\)-vector \(\mu\), and an \(S \times T\) filter \(\{a(u)\}\), so that

\[ \mu + \sum_{u=-\infty}^{\infty} a(t-u) X(u) , \quad (2.5) \]
Which is close to \( Y(t) \). Suppose we measure closeness by the \( S \times S \) Hermitian matrix

\[
E \left\{ \left[ (Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u))Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u) \right]^T \right\} \tag{2.6}
\]

**Theorem 2.1**

Consider an \( (r+s) \) vector-valued second-order of stability time series of the form (2.1) with mean (2.2) and autocovariance functions (2.3). Suppose \( c_{xs}(u), c_{ys}(u) \) are absolutely summable and suppose \( f_{xs}(\lambda), f_{ys}(\lambda) \) and \( f_{xs}(\lambda) \) are given by (2.4) and \( f_{xs}(\lambda) \) is nonsingular, \( -\infty < \lambda < \infty \).

Then the, \( \mu \), and \( a(u) \) that minimize (2.6) are given by

\[
\mu = c_y - \left( \sum_{u=-\infty}^{\infty} a(u) \right) c_y = c_y - A(0)c_y \tag{2.7}
\]

and

\[
a(u) = (2\pi)^{-1} \int_{0}^{2\pi} A(\alpha) \exp(iu\alpha) d\alpha \tag{2.8}
\]

where

\[
A(\lambda) = f_{ys}(\lambda) f_{xs}(\lambda)^{-1}
\]

the filter \( \{a(u)\} \) is absolutely summable. The minimum achieved is

\[
\int_{0}^{2\pi} \left[ f_{ys}(\alpha) - f_{ys}(\alpha) f_{xs}(\alpha)^{-1} f_{ys}(\alpha) \right] d\alpha \tag{2.10}
\]

where \( A(\lambda) \) is the transfer function of the \( S \times r \) filter achieving the indicated minimum. We call \( A(\lambda) \), the complex regression coefficient of \( Y(t) \) on \( X(t) \) at frequency \( \lambda \).

**Proof**

Let \( A(\lambda) \) be the transfer function of \( a(u) \) which defined as (2.8). We may write as,

\[
E \left\{ \left[ (Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u))Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u) \right]^T \right\}
\]

\[
= \text{cov}(Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)) + E[Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)]^T \times
\]

\[
E[Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)]^T
\]

\[
= E \left\{ \left[ (Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)) - E[Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)] \right]^T \right\}
\]

\[
\times \left\{ \left[ (Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)) - E[Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)] \right]^T \right\} +
\]

\[
+ E[Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)] \times E[Y(t) - \mu - \sum_{u=-\infty}^{\infty} a(t-u)X(u)]^T
\]

\[
= \int_{-\pi}^{\pi} \left[ f_{ys}(\alpha) - f_{ys}(\alpha) f_{xs}(\alpha)^{-1} f_{ys}(\alpha) \right] d\alpha +
\]

\[
+ \int_{-\pi}^{\pi} [A(\alpha) f_{xs}(\alpha) - f_{ys}(\alpha)] f_{xs}(\alpha)^{-1} \times
\]

\[
\int_{-\pi}^{\pi} [A(\alpha) f_{xs}(\alpha) - f_{ys}(\alpha)] f_{xs}(\alpha)^{-1} \times
\]

\[
\times [A(\alpha) f_{xs}(\alpha) - f_{ys}(\alpha)] d\alpha + [c_y - \mu - \sum_{u=-\infty}^{\infty} a(t-u)c_y] \times
\]

\[
\int_{-\pi}^{\pi} [f_{ys}(\alpha) - f_{ys}(\alpha) f_{xs}(\alpha)^{-1} f_{ys}(\alpha)] d\alpha
\]

let

\[
c_y - \mu - \sum_{u=-\infty}^{\infty} a(t-u)c_y = 0 \tag{2.10}
\]

then

\[
\mu = c_y - \sum_{u=-\infty}^{\infty} a(t-u)c_y = c_y - A(0)c_y
\]

and

\[
A(\alpha) f_{xs}(\alpha) - f_{ys}(\alpha) = 0
\]

\[
\Rightarrow A(\alpha) = f_{ys}(\alpha) f_{xs}(\alpha)^{-1}
\]

Using (2.7) and (2.8) the minimum achieved

\[
\int_{0}^{2\pi} \left[ f_{ys}(\alpha) - f_{ys}(\alpha) f_{xs}(\alpha)^{-1} f_{ys}(\alpha) \right] d\alpha
\]

**3. ASYMPTOTIC PROPERTIES OF EXPANDED FINITE FOURIER TRANSFORM WITH MISSED OBSERVATIONS**
Let $h_a^{(T)}(\lambda)$ be the discrete expanded finite Fourier transform which is defined as

$$h_a^{(T)}(\lambda) = \left[2\pi \sum_{a=1}^\infty d_a^{(T)}(t) \right] \left\{ \sum_{j=\min(r,s)}^{\infty} e^{-i\lambda j}, \quad -\infty < \lambda < \infty \right.$$(3.1)

where

$$\psi_a(t) = B_a(t)Z_a(t), \quad a = 1,2,\ldots, \min(r,s), \quad (3.2)$$

$X_a(t), Y_a(t)$ are the observations on the stability stochastic processes, $B_a(t)$ is Bernoulli sequence of random variable which is stochastically independent of $X_a(t), Y_a(t)$ which satisfies

$$B_a(t) = \begin{cases} 1, & \text{if } X_a(t), Y_a(t) \text{ are observed;} \\ 0, & \text{otherwise.} \end{cases} \quad (3.3)$$

Let $B_a(t)$ be an independent and identically distributed random variables with

$$P[B_a(t) = 1] = p_a, \quad P[B_a(t) = 0] = q_a, \quad (3.4)$$

where $p_a + q_a = 1$.

The data window function $d_a^{(T)}(t) = d_a^{(T)}(\frac{t}{T}), \quad t \in (0, T)$ is bounded has bounded variation and vanishes for all $t$ outside the interval $[0, T]$.

Assumption

Let $d_a^{(T)}(t), t \in R, a = 1, \ldots$ has bounded variation and vanishes for $t > T-1, t < 0$ then,

$$G_{a_1, \ldots, a_k} = \sum_{t=0}^{T-1} \left\{ \prod_{j=1}^k d_{a_j}^{(T)}(t) \right\} \exp\{-i\lambda t\},$$

For $-\infty < \lambda < \infty$ and $a_1, \ldots, a_k = 1,2,\ldots, r$. The following theorem will give the asymptotic properties of $\psi_a(t)$ which is defined as (3.2).

**Theorem 3.1**

Let $\psi_a(t) = B_a(t)Z_a(t), \quad a = 1,2,\ldots, \min(r,s)$ are missed observations on the stable stochastic processes, $X_a(t), Y_a(t), \quad a = 1,2,\ldots, \min(r,s)$ and $B_a(t)$ is Bernoulli sequence of random variables which satisfies equations$(3.1),(3.4)$, Then,

$$E(\psi_a(t)) = 0, \quad (3.5)$$

$$\text{Cov}[\psi_a(t_1), \psi_a(t_2)] = p_a \begin{bmatrix} c_0(u) & c_\infty(u) \\ c_\infty(u) & A(\alpha)c_\infty(u)A(\alpha)^T \end{bmatrix} \quad (3.6)$$

$$\text{Cov}[\psi_a(t_1), \psi_a(t_2)] = q_a \begin{bmatrix} \int_{-\infty}^{\infty} f_{\psi_a}(v) \exp(ivu)dv & \int_{-\infty}^{\infty} f_{\psi_a}(v) \exp(ivu)dvA(\alpha)^T \\ \int_{-\infty}^{\infty} f_{\psi_a}(v) \exp(ivu)dvA(\alpha) & \int_{-\infty}^{\infty} f_{\psi_a}(v) \exp(ivu)dvA(\alpha)^T \end{bmatrix} \quad (3.7)$$

**Proof**

Since $X(t)$ is a strictly stability series and $B_a(t)$ is independent of $Z_a(t)$ then (3.5) comes directly.

$$\text{Cov}[\psi_a(t_1), \psi_a(t_2)] =$$

$$= \text{Cov}[B_a(t)Z_a(t), B_a(t)Z_a(t)]$$

$$= \begin{bmatrix} B_{a_1}(t_1)X_{a_1}(t_1) & B_{a_1}(t_1)Y_{a_1}(t_1) \\ B_{a_2}(t_1)X_{a_2}(t_1) & B_{a_2}(t_1)Y_{a_2}(t_1) \end{bmatrix}$$

$$= \begin{bmatrix} B_{a_1}(t_1)X_{a_1}(t_1) & B_{a_1}(t_1)Y_{a_1}(t_1) \\ B_{a_2}(t_1)X_{a_2}(t_1) & B_{a_2}(t_1)Y_{a_2}(t_1) \end{bmatrix}$$

$$= \begin{bmatrix} p_{a_1} \text{Cov}[X_{a_1}(t_1), X_{a_1}(t_2)] & p_{a_1} \text{Cov}[X_{a_1}(t_1), Y_{a_1}(t_2)] \\ p_{a_2} \text{Cov}[X_{a_2}(t_1), X_{a_2}(t_2)] & p_{a_2} \text{Cov}[X_{a_2}(t_1), Y_{a_2}(t_2)] \end{bmatrix}$$

$$= \begin{bmatrix} p_{a_1} \text{Cov}[X_{a_1}(t_1), X_{a_1}(t_2)] & p_{a_1} \text{Cov}[X_{a_1}(t_1), Y_{a_1}(t_2)] \\ p_{a_2} \text{Cov}[X_{a_2}(t_1), X_{a_2}(t_2)] & p_{a_2} \text{Cov}[X_{a_2}(t_1), Y_{a_2}(t_2)] \end{bmatrix}$$

$$= \begin{bmatrix} p_{a_1}A(\alpha)C_{a_1}(t_1-t_2) & p_{a_2}A(\alpha)C_{a_2}(t_1-t_2) \\ p_{a_1}A(\alpha)C_{a_1}(t_1-t_2) & p_{a_2}A(\alpha)C_{a_2}(t_1-t_2) \end{bmatrix}$$

from the stability and the independence then,

$$\text{Cov}[\psi_a(t_1), \psi_a(t_2)] = p_{a_1} \begin{bmatrix} c_0(u) & c_\infty(u)A(\alpha)^T \\ c_\infty(u)A(\alpha) & A(\alpha)c_\infty(u)A(\alpha)^T \end{bmatrix}$$

and

$$\text{Cov}[\psi_a(t_1), \psi_a(t_2)] =$$
such that

\[
A \text{ and } \lambda \quad \text{be defined as}
\]

\[
\left\{ \begin{array}{c}
\int_{-\alpha}^{\alpha} f_{x,v}(v) \exp(\imath v u) \, dv \\
\int_{-\alpha}^{\alpha} f_{x,v}(v) \exp(\imath v u) \, dv
\end{array} \right\}
\]

\[
A(\alpha)
\]

\[
\left\{ \begin{array}{c}
\int_{-\alpha}^{\alpha} f_{x,v}(v) \exp(\imath v u) \, dv \\
\int_{-\alpha}^{\alpha} f_{x,v}(v) \exp(\imath v u) \, dv
\end{array} \right\}
\]

\[
A(\alpha)
\]

Definition: The complex normal distribution: Suppose X and Y are random vectors in \( R^k \) such that vec[X Y] is a 2k-dimensional normal vector. Then we say that the complex random vector \( Z = X + iY \) has the complex normal distribution. This distribution can be described with three parameters: 

\[
\mu = E(Z), \Gamma = E[(Z - \mu)(\overline{Z} - \overline{\mu})^T], \]

\[
C = E[(Z - \mu)(Z - \mu)^T],
\]

where \( Z^T \) denotes matrix transpose, and \( \overline{Z} \) denotes complex conjugate. Here the parameter \( \mu \) can be an arbitrary \( k \)-dimensional complex vector, the covariance matrix \( \Gamma \) must be Hermitian and non-negative definite; the relation matrix \( C \) should be symmetric. Moreover, matrices \( \Gamma \) and \( C \) are such that the matrix \( \overline{\Gamma} - C^T \Gamma^{-1} C \) is also non-negative definite. Matrices \( G \) and \( C \) are related to the covariance matrices of \( X \) and \( Y \) via expressions

\[
V_{xx} = E[(X - \mu_x)(X - \mu_x)^T] = \frac{1}{2} \text{Re}[\Gamma + C],
\]

\[
V_{yy} = E[(Y - \mu_y)(Y - \mu_y)^T] = \frac{1}{2} \text{Im}[\Gamma + C],
\]

\[
V_{xy} = E[(Y - \mu_y)(X - \mu_x)^T] = \frac{1}{2} \text{Im}[\Gamma - C],
\]

\[
V_{yx} = E[(X - \mu_x)(Y - \mu_y)^T] = \frac{1}{2} \text{Re}[\Gamma - C],
\]

and conversely

\[
\Gamma = V_{xx} + V_{yy} + i(V_{yx} - V_{xy}), \Gamma = V_{xx} - V_{yy} + i(V_{yx} - V_{xy}).
\]

Theorem 3.2

Let \( \psi_a(t) \) be missed observations on the stable stochastic process \( X_a(t), Y_a(t), a = 1, \ldots, \min(\ r, s) \) and \( B_a(t) \) is Bernoulli sequence of random variables which satisfies equations (3.3) and (3.4). Let \( h_a^{(T)}(\lambda) \) be defined as (3.1) and \( d_a^{(T)}(\lambda) \) satisfies assumption, then \( h_a^{(T)}(\lambda) \) will be distributed approximately as,

\[
h_a^{(T)}(\lambda) \equiv \frac{1}{p_{n_a}} \left[ \frac{1}{2} \text{Re}[\Gamma + C], \frac{1}{2} \text{Im}[\Gamma + C], \frac{1}{2} \text{Im}[\Gamma - C], \frac{1}{2} \text{Re}[\Gamma - C], \right]
\]

\[
\theta_a^{(T)}(\lambda) \equiv \frac{1}{p_{n_a}} \left[ \frac{1}{2} \text{Re}[\Gamma + C], \frac{1}{2} \text{Im}[\Gamma + C], \frac{1}{2} \text{Im}[\Gamma - C], \frac{1}{2} \text{Re}[\Gamma - C], \right]
\]
\[\begin{align*}
= (2\pi)^{-1} & \left[ G_{\alpha_2}^{(T)}(0) \right]^{-1} \sum_{t_1=0}^{T-1} d_{\alpha_1}^{(T)}(t_1) \sum_{t_2=0}^{T-1} d_{\alpha_2}^{(T)}(t_2) \times \\
\times \frac{1}{2\pi} \exp \left[ -i\omega t_1 + i\omega t_2 - i\omega t_1 - i\omega t_2 \right] & \left[ \int_{-\infty}^{\infty} f_{\alpha_1}(v) dv \right] \left[ \int_{-\infty}^{\infty} f_{\alpha_2}(v) dv \right] \frac{1}{2\pi} \\
= (2\pi)^{-1} & \left[ G_{\alpha_2}^{(T)}(0) \right]^{-1} \sum_{t_1=0}^{T-1} d_{\alpha_1}^{(T)}(t_1) \sum_{t_2=0}^{T-1} d_{\alpha_2}^{(T)}(t_2) \times \\
\times \frac{1}{2\pi} \exp \left[ -i(\lambda_1 - \lambda) t_1 - i(\lambda_2 - \lambda) t_2 \right] & \left[ \int_{-\infty}^{\infty} f_{\alpha_1}(v) dv \right] \left[ \int_{-\infty}^{\infty} f_{\alpha_2}(v) dv \right] \\
= \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} & (3.11)
\end{align*}\]

where

\[\begin{align*}
\beta_1 &= p_{\alpha_1} \int_{-\infty}^{\infty} f_{\alpha_1}(v) \left( (2\pi)^{-1} \left[ G_{\alpha_2}^{(T)}(0) \right]^{-1} \sum_{t_1=0}^{T-1} d_{\alpha_1}^{(T)}(t_1) \sum_{t_2=0}^{T-1} d_{\alpha_2}^{(T)}(t_2) \right) \\
& \times \exp \left[ -i(\lambda_1 - \lambda) t_1 - i(\lambda_2 - \lambda) t_2 \right] dv \\
& = p_{\alpha_1} \int_{R} f_{\alpha_1}(v) \Omega_{\alpha_2}^{(T)}(\lambda_1 - v, \lambda_2 - v) dv,
\end{align*}\]

similarly

\[\begin{align*}
\beta_2 &= p_{\alpha_1} \int_{R} f_{\alpha_1}(v) A(\alpha)^{T} \Omega_{\alpha_2}^{(T)}(\lambda_1 - v, \lambda_2 - v) dv,
\beta_3 &= p_{\alpha_1} \int_{R} A(\alpha) f_{\alpha_1}(v) \Omega_{\alpha_2}^{(T)}(\lambda_1 - v, \lambda_2 - v) dv,
\end{align*}\]

and

\[\beta_4 = p_{\alpha_1} \int_{R} A(\alpha) f_{\alpha_1}(v) A(\alpha)^{T} \Omega_{\alpha_2}^{(T)}(\lambda_1 - v, \lambda_2 - v) dv\]

Now from equation (3.10) and (3.11) then equation (3.8) is obtained which complete the proof.

From equation (3.11) we can drive the following corollary by putting \(\lambda_1 = \lambda_2 = \lambda\), \(\lambda_1, \lambda_2, \lambda \in R\).

**Corollary 3.1**

let \(h_{\alpha}^{(T)}(\lambda), a = 1, 2, \ldots, \min(r, s), \lambda \in R\) be defined as (3.1), then the dispersion of \(h_{\alpha}^{(T)}(\lambda)\) satisfies the following property:

\[Dh_{\alpha}^{(T)}(\lambda) = p_{\alpha} \left[ \int_{-\infty}^{\infty} f_{\alpha}(v) \Omega_{\alpha_2}^{(T)}(\lambda - v) dv \right] \left[ \int_{-\infty}^{\infty} A(\alpha) f_{\alpha}(v) \Omega_{\alpha_2}^{(T)}(\lambda - v) dv \right].\]

and

\[\Omega_{\alpha_2}^{(T)}(\lambda) = (2\pi)^{-1} \left[ G_{\alpha_2}^{(T)}(0) \right]^{-1} \left| G_{\alpha_2}^{(T)}(\lambda) \right|^2 ,\]

where \(G_{\alpha_2}^{(T)}(\lambda), a = 1, 2, \ldots, \min(r, s), \lambda \in R\) be defined in Assumption.

**Proof**

From equation (3.11), we get

\[Dh_{\alpha}^{(T)}(\lambda) = p_{\alpha} \left[ \int_{-\infty}^{\infty} f_{\alpha}(v) \Omega_{\alpha_2}^{(T)}(\lambda - v) dv \right] \left[ \int_{-\infty}^{\infty} A(\alpha) f_{\alpha}(v) \Omega_{\alpha_2}^{(T)}(\lambda - v) dv \right].\]

When \(\lambda_1 = \lambda_2 = \lambda, \lambda \in R\) and \(a_1 = a_2 = a, a = 1, \ldots, \min(r, s)\).

By putting \(\lambda - v = \gamma\), then formula (3.12) is obtained.

**Theorem 3.3**

For any \(\lambda \in R\), the function \(\Omega_{\alpha_2}^{(T)}(\lambda), a = 1, \ldots, \min(r, s)\) is the kernel that satisfies the following properties:

1. \(\int_{-\infty}^{\infty} \Omega_{\alpha_2}^{(T)}(\lambda) d\lambda = 1, a = 1, \ldots, \min(r, s), \lambda \in R\) \hspace{1cm} (3.13)
2. \(\lim_{T \to \infty} \int_{-\infty}^{\infty} \Omega_{\alpha_2}^{(T)}(\lambda) d\lambda = \lim_{T \to \infty} \int_{-\infty}^{\infty} \Omega_{\alpha_2}^{(T)}(\lambda) d\lambda = 0, \lambda \in R\) \hspace{1cm} (3.14)
3. \(\lim_{T \to \infty} \int_{-\infty}^{\infty} \Omega_{\alpha_2}^{(T)}(\lambda) d\lambda = 1, \lambda \in R\) \hspace{1cm} (3.15)

**Theorem 3.4**

If the spectral density function \(f_{\alpha}(X), a = 1, \ldots, \min(r, s)\), \(X \in R\) is bounded continuous at a point \(X = \lambda, \lambda \in R\) and the function \(\Omega_{\alpha_2}^{(T)}(X), a = 1, \ldots, \min(r, s)\), \(X \in R\) satisfies the properties of theorem 3.3, then,
\[
\lim_{T \to \infty} D_h^{(T)}(\lambda) = p_{aa} \left[ \begin{array}{cc}
f_{aa}(\lambda) & f_{aa}(\lambda)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda) & A(\alpha)f_{aa}(\lambda)A(\alpha)^T
\end{array} \right]
\]

\[
, \quad a = 1, \ldots, \min(r, s).
\]

Since \( f_{aa}(\gamma) \) is continuous at a point \( \gamma = \lambda \), \( a_1, a_2 = 1, \ldots, \min(r, s) \), then we get
\[
J_2 = p_{aa} \int_{\lambda - \delta}^{\lambda + \delta} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

\[
- \left[ \begin{array}{cc}
f_{aa}(\lambda) & f_{aa}(\lambda)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda) & A(\alpha)f_{aa}(\lambda)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

\[
\leq \int_{\lambda - \delta}^{\lambda + \delta} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

\[
\leq \int_{-\infty}^{\infty} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

\[
\leq \int_{-\infty}^{\infty} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

\[
\leq \int_{-\infty}^{\infty} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

\[
\leq \int_{-\infty}^{\infty} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

Hence, \( J_2 \leq \varepsilon \). Now \( J_2 \) is very small according to any \( \varepsilon \) is very small, consequently \( J_2 = 0 \). Suppose that \( f_{aa}(\lambda) = a = 1, \ldots, \min(r, s) \), \( \lambda \in \mathbb{R} \) is bounded by a constant \( M \), then
\[
J_1 \leq 2M \int_{-\infty}^{\infty} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

\[
\leq \int_{-\infty}^{\infty} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

According to property (3.14), similarly \( J_3 \to 0 \), therefore,
\[
\lim_{T \to \infty} \left[ \begin{array}{cc}
f_{aa}(\lambda - \gamma) & f_{aa}(\lambda - \gamma)A(\alpha)^T \\
A(\alpha)f_{aa}(\lambda - \gamma) & A(\alpha)f_{aa}(\lambda - \gamma)A(\alpha)^T
\end{array} \right] \Omega_a^{(T)}(\gamma) d\gamma
\]

which completes the proof of the theorem.

**Lemma 3.1**

If the data window function \( d_a^{(T)}(t), t \in \mathbb{R} \), \( a = 1, r \) is bounded and has bounded variations and equal zero outside the interval \([0, T - 1]\); then
\[
\sum_{t=0}^{T-1} d_a^{(T)}(t) \sim T \int_{0}^{T} d_a^{(T)}(u) du,
\]

where,
\[
\frac{1}{T} \sum_{t=0}^{T-1} d_a^{(T)}(t) \longrightarrow \frac{1}{T} \int_{0}^{T} d_a^{(T)}(u) du, \quad a = 1, r, T = 1, 2, \ldots
\]

**Lemma 3.2**

\[
\frac{1}{T} \sum_{t=0}^{T-1} d_a^{(T)}(t) \longrightarrow \frac{1}{T} \int_{0}^{T} d_a^{(T)}(u) du, \quad a = 1, r, T = 1, 2, \ldots
\]
Suppose \( d^{(T)}_a(t), t \in R, \ a = 1, r \) is bounded by a constant \( L \) and satisfying the Lipschitz condition,

\[
\sum_{u=0}^{T-1} \left| d^{(T)}_a(t+u) - d^{(T)}_a(t) \right| < \infty,
\]

\[
- \sum_{t=0}^{T-1} d^{(T)}_{a_1}(t) d^{(T)}_{a_2}(t) \exp \left\{ -i \lambda t \right\} \leq \varepsilon |u|,
\]

(3.19)

then,

\[
\sum_{t=0}^{T-1} d^{(T)}_a(u+t) d^{(T)}_a(t) \exp \left\{ -i \lambda t \right\} \leq L \varepsilon |u|,
\]

(3.20)

for all constant \( \varepsilon, \ u = [-T - 1], \ (T - 1) \) and \( \lambda \in [-\pi, \pi] \).

**Lemma 3.3**

For all \( \lambda_1, \lambda_2 \in [-\pi, \pi], \ (\lambda_1 - \lambda_2) \neq \text{(mod 2}\pi) \) and \( d^{(T)}_a(t), t \in R, \ a = 1, ..., \min(r, s) \) is bounded by a constant \( L \) and satisfying Lipschitz condition (3.19), then,

\[
\text{Cov} \left[ h^{(T)}_{a_1}(\lambda_1), h^{(T)}_{a_2}(\lambda_2) \right] \leq \frac{L \varepsilon}{2\pi} \sum_{u=0}^{T-1} \left| \frac{d^{(T)}_{a_1}(\lambda_1)}{d^{(T)}_{a_2}(\lambda_2)} \right|^2 \times
\]

\[
\times \left\{ \frac{1}{L_0(\lambda_1 - \lambda_2)^2} \sum_{t=0}^{T-1} \left| C_{a_1a_2}(u) \right| + \sum_{u=0}^{T-1} \left| C_{a_1a_2}(u) \right| \right\},
\]

(3.21)

for all \( a_1, a_2 = 1, ..., \min(r, s) \).

**Theorem 3.5**

For all \( \lambda_1, \lambda_2 \in [-\pi, \pi], \ (\lambda_1 - \lambda_2) \neq \text{(mod 2}\pi) \) and \( d^{(T)}_a(t), t \in R, \ a = 1, ..., \min(r, s) \) is bounded and

\[
\sum_{t=0}^{T-1} \left| C_{a_1a_2}(u) \right| < \infty,
\]

(3.22)

then

\[
\lim_{T \to \infty} \text{Cov} \left[ h^{(T)}_{a_1}(\lambda_1), h^{(T)}_{a_2}(\lambda_2) \right] = 0,
\]

(3.23)

for all \( a_1, a_2 = 1, ..., \min(r, s) \).

The proof comes directly from Lemma 3.3 and Lemma 3.1.

### 4. APPLICATIONS

We will apply our theoretical study in two cases in climate and economy as in the following sections.

#### 4.1. Studying the temperature and solar radiation

The data manipulated in this research make up a monthly chronic series that represents the average of the monthly temperature and solar radiation in Tripoli in Libya. The data is extracted from the meteorological center of Tripoli, for the period from January 2005 to December 2013.

##### 4.1.1. Studying the temperature

In this study we will comparison between our results, model of strictly stability time series (temperature) with some missing observations and the classical results, where all observations are available.

Let \( \Phi_a(t) = B_a(t)X_a(t) , a = 1, 2, ..., r \), where \( X_a(t) (t = 0, 1, 2, ..., T) \) be a strictly stability r-vector valued time series and \( B_a(t) \) is Bernoulli sequence of independent random variable of \( X_a(t) \) which satisfies equations (3.3) and (3.4), we suppose that the data \( X_a(t), t = 1, 2, ..., T \) is the average of the monthly temperature, where all observations are available, \( B = 1, \ \Phi_a(t) = X_a(t) \), which is the classical case suppose that there is some missing observations in a random way, i.e., \( B = 0 \), table 4.1.1 shows the comparison of these results with and without missed observations.

**Table 4.1.1:** The comparison of the results with and without missed observations

<table>
<thead>
<tr>
<th>Series without missed observations</th>
<th>Series with missed observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Series Plot of Temperature</td>
<td>Time Series Plot of Testi</td>
</tr>
</tbody>
</table>

The average monthly temperatures
Let \( \phi_a(t) = B_a(t)Y_a(t) \), \( a = 1, 2, \ldots, s \), where \( Y_a(t) \), \( t = 0, \pm 1, \ldots \), be a strictly stability \( s \)-vector valued time series and \( B_a(t) \) is Bernoulli sequence of random variable which is stochastically independent of \( Y_a(t) \) which satisfies equations (3.3) and (3.4), we suppose that the data \( Y_a(t) \), \( t = \{1, 2, \ldots, T\} \) is the average of the monthly temperature, where all observations are available, \( \bar{Y} = \frac{1}{T} \), \( \phi_a(t) = Y_a(t) \), which is the classical case, suppose that there is some missing observations in a random way, i.e., \( \bar{B} = 0 \), Table 4.1.2 shows the comparison of these results with and without missed observations.

### Table 4.1.2: The comparison of the results with and without missed observations of the solar radiation

<table>
<thead>
<tr>
<th>Series without missed observations</th>
<th>Series with missed observations</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="The average monthly solar radiation" /></td>
<td><img src="image2" alt="The average monthly solar radiation" /></td>
</tr>
<tr>
<td><img src="image3" alt="ACF of the seasonal difference" /></td>
<td><img src="image4" alt="ACF of the seasonal difference" /></td>
</tr>
<tr>
<td><img src="image5" alt="PACF of the seasonal difference" /></td>
<td><img src="image6" alt="PACF of the seasonal difference" /></td>
</tr>
</tbody>
</table>

### 4.1.2 Studying the solar radiation

In this study we will comparison between our results, model of strictly stability time series (Solar Radiation) with some missing observations and the classical results, where all observations are available.
AR 3 -0.3299 1.0004 -3.29 0.001 AR 3 -0.4197 0.0976 -4.33 0.000
SMA 12 0.9057 0.1119 8.09 0.000 SMA 12 1.0427 0.1089 9.58 0.000
SMA 24 -0.0570 0.1637 -0.35 0.729 SMA 24 -0.2015 0.1577 -1.28 0.205
Constant -0.1524 0.8091 -0.19 0.851 Constant -0.0518 0.7478 -0.07 0.945
Differencing: 0 regular, 1 seasonal of order 12 Differencing: 0 regular, 1 seasonal of order 12
Number of observations: Original series 108, after differencing 96 Number of observations: Original series 108, after differencing 96
Residuals: SS = 113197 (back forecasts excluded) Residuals: SS = 122927 (back forecasts excluded)
MS = 1258 DF = 90 MS = 1366 DF = 90
Modified Box-Pierce (Ljung-Box) Chi-Square statistic Modified Box-Pierce (Ljung-Box) Chi-Square statistic
Lag 12 24 36 48 Lag 12 24 36 48
Chi-Square 10.0 16.0 35.1 44.7 Chi-Square 8.5 21.3 41.1 51.2
DF 6 18 30 42 DF 6 18 30 42
P-Value 0.123 0.589 0.240 0.358 P-Value 0.206 0.266 0.085 0.15

4.1.3. Studying The Regression Between Solar Radiation And Temperature

In this section we adjust the regression model which represents the relationship between Monthly rate of solar radiation in watt /m^2 rate and the average monthly temperature in the period from 2005 to 2013.

In this study we will compare between our results with some missing observations and the classical results where all observations are available.

Let \( Z(t) = [X(t) \ Y(t)] \) where \( X(t) \) is the series of average of temperature and \( Y(t) \) is the series of the average of solar radiation, first we consider that the observations are available \( P = 1, \psi(t) = B(t)Z(t) = pZ(t) = Z(t) \), then consider that there are some missing of observations randomly, \( P = 0 \). We used SPSS,MINITAB to investigate our results which is shown in table 4.1.3.

Table - 4.1.3: The comparison of the results with and without missed observations of the regression analysis

<table>
<thead>
<tr>
<th>Without missed observations</th>
<th>With missed observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>The regression equation is</td>
<td>The regression equation is</td>
</tr>
<tr>
<td>solar radiation = -10.4 + 12.7 temperature</td>
<td>Solar = -9.9 + 12.7 Temperature</td>
</tr>
<tr>
<td>Predictor Coef SE Coef T P</td>
<td>Predictor Coef SE Coef T P</td>
</tr>
<tr>
<td>Constant -10.42 22.36 -0.47 0.642</td>
<td>Constant -9.94 22.71 -0.44 0.663</td>
</tr>
</tbody>
</table>

4.1.4. Conclusion

1. Tables 4.1.1 and 4.1.2 shows the study of time series with missed observations and the original time series and we investigated that they have the same results.
2. Table 4.1.3 shows the study of regression model between Monthly average of solar radiation and average monthly temperature with some missed observations which had the same results of the study of the classical regression model.

4.2. Studying the Export and the Gross domestic product

The data manipulated in this research make up chronic series that represents the Export and the Gross domestic product. The data is extracted from the Central Bank of Libya for the period from 1970 to 2012.

4.2.1. Studying the Export

In this study we will comparison between our results, model of strictly stability time series (Export) with some missing observations and the classical results, where all observations are available.

Let \( \Phi_i(t) = B(t)X(t), a = 1, 2, \ldots, r \), where \( X_a(t), (t = 0, \pm 1, \ldots) \) be a strictly stability r-vector valued time series and \( B_a(t) \) is
Bernoulli sequence of independent random variable of \( X_a(t) \) which satisfies equations (3.3) and (3.4), we suppose that the data \( X_a(t), t = 1,2, \ldots, T \) is the Export, where all observations are available, \( B = 1, \Phi_a(t) = X^t \), which is the classical case suppose that there is some missing observations in a random way, i.e., \( B = 0 \), table 4.2.1 shows the comparison of these results with and without missed observations.

Table -4.2.1: The comparison of the results with and without missed observations

<table>
<thead>
<tr>
<th>Type</th>
<th>Coef</th>
<th>SE Coef</th>
<th>T</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR 1</td>
<td>-1.1606</td>
<td>0.1112</td>
<td>-10.44</td>
<td>0.000</td>
</tr>
<tr>
<td>MA 1</td>
<td>0.0554</td>
<td>0.2152</td>
<td>0.26</td>
<td>0.798</td>
</tr>
<tr>
<td>Constant</td>
<td>-197</td>
<td>1271</td>
<td>-0.15</td>
<td>0.878</td>
</tr>
<tr>
<td>Differenting: 2 regular differences</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.2.2. Studying the Gross domestic product

In this study we will comparison between our results, model of strictly stability time series (Gross domestic product) with some missing observations and the classical results, where all observations are available.

Let \( \phi_d(t) = B_d(t)Y_a(t), a = 1,2, \ldots, s \), where \( Y_a(t), t = 1,2, \ldots, T \) be a strictly stability s-vector valued time series and \( B_d(t) \) is Bernoulli sequence of random variable which is stochastically independent of \( Y_a(t) \) which satisfies equations (3.3) and (3.4), we suppose that the data \( Y_a(t), t = 1,2, \ldots, T \) is the Gross domestic product, where all observations are available, \( B = 1, \phi_d(t) = Y_a(t) \), which is the classical case, suppose that there is some missing observations in a random way, i.e., \( B = 0 \), table 4.2.2 shows the comparison of these results with and without missed observations.

Table-4.2.2 The comparison of the results with and without missed observations of the Gross domestic product

<table>
<thead>
<tr>
<th>Type</th>
<th>Coef</th>
<th>SE Coef</th>
<th>T</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR 1</td>
<td>-1.1559</td>
<td>0.1081</td>
<td>-10.69</td>
<td>0.000</td>
</tr>
<tr>
<td>MA 1</td>
<td>-0.0757</td>
<td>0.2246</td>
<td>-0.34</td>
<td>0.738</td>
</tr>
<tr>
<td>Constant</td>
<td>776</td>
<td>1596</td>
<td>0.49</td>
<td>0.630</td>
</tr>
<tr>
<td>Differenting: 2 regular differences</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of observations: Original series 43, after differencing 41
4.2.3. Studying the regression between Gross domestic product and Export

In this section we adjust the regression model which represents the relationship between the Gross domestic product and Export in the period from 1970 to 2012 million Libyan dinars.

In this study we will comparison between our results with some missing observations and the classical results where all observations are available.

Let \( Z(t) = [X(t) \ Y(t)]^T \) where \( X(t) \) is the series of the Export average and \( Y(t) \) is the series of the Gross domestic product, first we consider that the observations are available \( P = 1 \), \( \psi(t) = B(t)Z(t) = pZ(t) = Z(t) \), then consider that there are some missing of observations randomly, \( P = 0 \). We used SPSS, MINITAB to investigate our results which is shown in table 4.2.3.

Table 4.2.3: The comparison of the results with and without missed observations of the regression analysis

<table>
<thead>
<tr>
<th>Without missed observations</th>
<th>With missed observations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Regression Analysis</strong>: ( Y(t) ) versus ( X(t) )</td>
<td><strong>Regression Analysis</strong>: ( Y^<em>(t) ) versus ( X^</em>(t) )</td>
</tr>
<tr>
<td>The regression equation is</td>
<td>The regression equation is</td>
</tr>
<tr>
<td>( Y(t) = 4029 + 1.52 ) Export ( X(t) )</td>
<td>( Y^<em>(t) = 4013 + 1.55 ) Export ( X^</em>(t) )</td>
</tr>
<tr>
<td>Predictor Coef SE Coef T P</td>
<td>Predictor Coef SE Coef T P</td>
</tr>
<tr>
<td>Constant 4029.2 724.2 5.56 0.000</td>
<td>Constant 4012.5 807.6 4.97 0.000</td>
</tr>
<tr>
<td>Export ( X(t) ) 1.52488 0.02872 53.10 0.000</td>
<td>Export ( X^*(t) ) 1.5544 0.03273 47.49 0.000</td>
</tr>
<tr>
<td>( S = 4004.51 ) R-Sq = 98.6% R-Sq(adj) = 98.5%</td>
<td>( S = 4449.61 ) R-Sq = 98.2% R-Sq(adj) = 98.2%</td>
</tr>
</tbody>
</table>

Analysis of Variance

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression 45207962941 45207962941</td>
<td>1</td>
<td>1219.14</td>
<td>1219.14</td>
</tr>
<tr>
<td>Residual Error 6577479499 16036085</td>
<td>42</td>
<td>45865442435</td>
<td>10910180</td>
</tr>
<tr>
<td>Total</td>
<td>43</td>
<td>44464419639</td>
<td>44464419639</td>
</tr>
</tbody>
</table>

4.2.4. Conclusion

1. Tables 4.2.1 and 4.2.2 shows the study of time series with missed observations and the original time series and we investigated that they have the same results.
2. Table 4.2.3 shows the study of regression models between Gross domestic product and Export with some missed observations which had the same results of the study of the classical regression models.
REFERENCES


