OBSERVATIONS ON $X^2 + Y^2 = 2Z^2 - 62W^2$

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Abstract - ***The Quadratic equation with four unknowns of the form $X^2 + Y^2 = 2Z^2 - 62W^2$ has been studied for its non-trivial distinct integral solutions. A few interesting relations among the solutions and special polygonal numbers are presented.***

Key Words: **Quadratic equation with four unknowns, Integral solutions.**

NOTATIONS:

$T_{3,n} = \frac{n(n+1)}{2}$ = Triangular number of rank n.

$T_{7,n} = \frac{n(5n-3)}{2}$ = Heptagonal number of rank n.

$T_{10,n} = n(4n-3)$ = Decagonal number of rank n.

$T_{12,n} = n(5n-4)$ = Dodecagonal number of rank n.

$T_{13,n} = \frac{n(11n-9)}{2}$ = Tridecagonal number of rank n.

$T_{17,n} = \frac{n(15n-13)}{2}$ = Heptadecagonal number of rank n.

$T_{18,n} = n(8n-7)$ = Octadecagonal number of rank n.

$Gno_n = (2n-1)$ = Gnomonic number of rank n.

INTRODUCTION:

In [1 to 12] some special types of quadratic equations with four unknowns have been analyzed for their non-trivial integral solutions. This communication concerns with another interesting quadratic equation with four variables represented by $X^2 + Y^2 = 2Z^2 - 62W^2$. To start with, we observe that the following non-zero quadruples (2rs-1, 2rs-1, r^2+s^2, r^2-s^2), (r^2-s^2-1, r^2-s^2-1,2rs, r^2+s^2), (y+2, y, y+2, ±1), (-1, y, -1, ±1), (-1, y, ±1, -1), (-y, y, ±1, -y) satisfy the equation under consideration. In the above quadruples, any two values of the unknowns are the same. In [13], the Quadratic equation with four unknowns of the form XY+X+Y+1 = $Z^2 - W^2$ has been studied for its non-trivial distinct integral solutions. Thus, towards this end we search for non-zero distinct integral solutions of the equation under consideration. A few interesting relations among the solutions are presented.

METHOD OF ANALYSIS:

The Quadratic Diophantine equation with four unknowns under consideration is

$$X^2 + Y^2 = 2Z^2 - 62W^2 \quad (1)$$

The substitution of the linear transformations

$$X = u + v,\ Y = v - u \text{ and } W = u \quad (2)$$

in (1) leads to

$$Z^2 = 32u^2 + v^2 \quad (3)$$

Four different choices of solutions to (3) are presented below. Once the values of u and v are known , using (2), the corresponding values of X and Y are obtained.

PATTERN 1:

In (3), $v^2 + 32u^2 = Z^2$

The solutions of the above equation is of the form

$$Z = 32A^2 + B^2$$
$$u = 2AB \qquad (4)$$
$$v = 32A^2 - B^2$$

Substituting the values of u and v in (2) , we get

$$X = X(A,B) = 32A^2 + 2AB - B^2$$
$$Y = Y(A,B) = 2AB - 32A^2 + B^2 \qquad (5)$$
$$Z = Z(A,B) = 32A^2 + B^2$$
$$W = W(A,B) = 2AB$$

Thus (5) gives the distinct integral solution of (1)

OBSERVATIONS:

1. $W\left(n^2+1,8\right) - Y\left(n,1\right) - 34 = 2T_{17,n} + 6T_{13,n} + 19(Gno_n)$

2. $X(A,2A-1) + Y(A,2A-1) = 2T_{10,A} + Gno_A + 1$

PATTERN 2:

(3) can be written as

$$Z^2 .1 = 32u^2 + v^2$$

Assuming $Z = 32p^2+q^2$ and write

$$1 = \left(\frac{(\sqrt{32}+i2)(\sqrt{32}-i2)}{36}\right) \qquad (6)$$

Substituting (6) in (3), using the method of factorization we get

$$\frac{\left(\sqrt{32}a+ib\right)^2\left(\sqrt{32}-ib\right)^2\left(\sqrt{32}+i2\right)\left(\sqrt{32}-i2\right)}{36} = \left(\sqrt{32}u+iv\right)\left(\sqrt{32}u-iv\right)$$

Now define,

$$(\sqrt{32}u+iv) = \frac{\left(\sqrt{32}a+ib\right)^2\left(\sqrt{32}+i2\right)}{6} \qquad (7)$$

$$(\sqrt{32}u-iv) = \frac{\left(\sqrt{32}a-ib\right)^2\left(\sqrt{32}-i2\right)}{6}$$

Equating the real and imaginary parts in (7), we have

$$u = \frac{1}{6}\left(32a^2 - 4ab - b^2\right)$$
$$v = \frac{1}{6}\left(64a^2 - 64ab - 2b^2\right)$$

Since our interest is on finding integer solutions we have choose

a and b suitably so that u and v are integers.

$$u = 32A^2 - 4AB - B^2 \qquad (8)$$
$$v = 64A^2 + 64AB - 2B^2$$

Thus, using the values of u and v and performing

a few calculations the values of X, Y and Z are obtained as follows:

$$X = X(A,B) = 96A^2 + 60AB - 3B^2$$
$$Y = Y(A,B) = 32A^2 + 68AB - B^2 \qquad (9)$$
$$W = W(A,B) = 32A^2 - 4AB - B^2$$
$$Z = Z(A,B) = 192A^2 + 6B^2$$

Thus (9) represents the non-trivial solution integral solution of (1)

OBSERVATIONS:

1. $X(A,1) - Y(A,1) = 8T_{18,A} + 24(Gno_A) - 22$

2. $Z(1,n) - W(1,2n-1) - T_{12,n} - 2T_{7,n} - 11n = 157$

PATTERN 3:

(3) can be written as

$$Z^2 - 32u^2 = v^2 \qquad (10)$$

Assuming $Z = a^2 - 32b^2$ and

$$1 = \left(\frac{(6+\sqrt{32})(6-\sqrt{32})}{4} \right) \qquad (11)$$

Substituting (11) in (10) , using the method of factorization

$$\frac{\left(a+\sqrt{32}b\right)^2\left(a-\sqrt{32}b\right)^2\left(6+\sqrt{32}\right)\left(6-\sqrt{32}\right)}{4} = \left(Z+\sqrt{32}u\right)\left(Z-\sqrt{32}u\right)$$

Now define,

$$(Z+\sqrt{32}u) = \frac{\left(a+\sqrt{32}b\right)^2\left(6+\sqrt{32}\right)}{2} \qquad (12)$$
$$(Z-\sqrt{32}u) = \frac{\left(a-\sqrt{32}b\right)^2\left(6-\sqrt{32}\right)}{2}$$

Equating the like terms in (12), we get

$$Z = \frac{1}{2}\left(6a^2+64ab+192b^2\right)$$
$$u = \frac{1}{2}\left(a^2+12ab+32b^2\right)$$

Since our interest is on finding integer solutions we have choose a and b suitably so that u and Z are integers.

$$Z = 6A^2+64AB+192B^2$$
$$u = A^2+12AB+32B^2 \qquad (13)$$
$$v = 2A^2-64B^2$$

Thus, using the values of u and v and performing a few calculations the values of X, Y and Z are obtained as follows:

$$X = X(A,B) = 3A^2+12AB-32B^2$$
$$Y = Y(A,B) = A^2-12AB-96B^2 \qquad (14)$$
$$W = W(A,B) = A^2+12AB+32B^2$$
$$Z = Z(A,B) = 6A^2+64AB+192B^2$$

Thus (14) represents the non-trivial solution integral solution of (1)

OBSERVATIONS:

1. $W(2A+1,1)+Y(2A+1,1)+95 = 16T_{3,A}$

2. When B=1, X+Y+Z+192 is a Nasty number.

PATTERN 4:

(3) can be written as

$$32u^2+v^2 = Z^2$$
$$32u^2 = Z^2-v^2$$
$$32u^2 = (Z+v)(Z-v)$$
$$(32u)(u) = (Z+v)(Z-v)$$
$$\frac{Z+v}{32u} = \frac{u}{Z-v} = \frac{A}{B}$$
$$(Z+v)B = 32uA \qquad (15)$$
$$(Z-v)A = uB \qquad (16)$$

Solving the equations (15) and (16) we get the solutions

$$Z = -B^2-32A^2$$
$$v = -32A^2+B^2$$
$$u = -2AB$$

Thus, using the values of u and v and performing a few calculations the values of X, Y and Z are obtained as follows:

$$X = -32A^2-2AB+B^2$$
$$Y = -32A^2+2AB+B^2 \qquad (17)$$
$$W = -2AB$$
$$Z = -32A^2-B^2$$

Thus (17) represents the non-trivial solution integral solution of (1).

OBSERVATIONS:

1. $Y(A,A)-X(A,A)-W(A,A)$ is a nasty number.

2. $Y(A,A) - X(A,A)$ is a square number.

3. $X(A,1) + W(A,1) + 2Gno_A = Z(A,1)$

4. $Y(A,1) = W(2A^2, A) + Z(A,1) + 6O_A + 2$

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BIOGRAPHIES

Dr.G.Janaki received the Ph.D., M.Sc., and M.Phil., degree in mathematics from Bharathidasan University, Trichy, South India. She completed her Ph.D. degree National College, Bharathidasan University. She has published many papers in international and national level journals. Her research area is "Number Theory".

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