AN ANALYTICAL SOLUTION OF POROUS MEDIUM EQUATION BY SIMILARITY TRANSFORMATION

Megha K. Tailor, S. S. Patel

GIDC Degree Engineering College ,
Abram, Navsari, ASH Department.

Abstract - The present paper analytically discusses the phenomenon of imbibition in double phase flow through porous media by using similarity technique. A possible analytically form for the saturation distribution in the investigated problem. For the particular case, when the saturation coefficient is regarded as a constant the displacing phase saturation distribution in terms of hypergeometric function have been obtained.

Key Words: **Porous media equation, Similarity transformation method, Hypergeometric Series**

1. INTRODUCTION

It is known that if a porous medium filled with some fluid is brought in two contacts with another fluid, which preferentially wets the medium, then there is a spontaneous flow of the wetting fluid into the medium. This phenomenon is called imbibition [7]. Many authors have investigated these problem from different viewpoints; for example, imbibition by Graham and Richardson (1959)[3]. Rijik (1960)[5] and verma (1969a)[8].

1.1 THEORETICAL FORMULATION OF THE PROBLEM

For the mathematical formulation of the above model we assume that the flow is governed by Darcy's law and consider only the statistical behavior of fingers. Thus the seepage velocities of liquid I and N are expressed as

$$V_i = \frac{K_i}{\gamma_i} K \frac{\partial p_i}{\partial x} \qquad \text{.... (2.1)}$$

$$V_n = \frac{K_n}{\gamma_n} K \frac{\partial p_n}{\partial x} \qquad \text{.... (2.2)}$$

The symbols are defined in the nomenclature. It may be mentioned that the statistical treatment of fingers (Verma 1969a) is formally identical to the Buckley-Leverett description of two immiscible fluids flow and the displacing phase saturation is defined by the average cross-sectional area occupied by fingers.

The condition of linear counters- current imbibition, viz. $V_i = -V_n$ yields

$$\frac{K_i}{\gamma_i}\frac{\partial p_i}{\partial x} + \frac{K_n}{\gamma_n}\frac{\partial p_n}{\partial x} = 0 \qquad \text{.... (2.3)}$$

The capillary pressure (p_c) is defined as the pressure discontinuity between the following phase and may be written as $p_c = p_n - p_i$ .... (2.4)

Eliminating p_n between eqns. (2.3) and (2.4), we get

$$\left[\frac{K_i}{\gamma_i} + \frac{K_n}{\gamma_n}\right]\frac{\partial p_i}{\partial x} + \frac{K_n}{\gamma_n}\frac{\partial p_c}{\partial x} = 0 \qquad \text{..... (2.5)}$$

Combining eqns. (2.1) and (2.5), we have

$$V_i = K \frac{\frac{K_i}{\gamma_i}\frac{K_n}{\gamma_n}}{\frac{K_i}{\gamma_i} + \frac{K_n}{\gamma_n}} \frac{\partial p_c}{\partial x} \qquad \text{.... (2.6)}$$

We may write the equation of continuity for injected water as

$$p\frac{\partial s_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0 \qquad \text{.... (2.7)}$$

Substituting the value of V_i from eqn. (2.6) into (2.7), we obtain,

$$p\frac{\partial s_i}{\partial t} + \frac{\partial}{\partial x}\left[\frac{KK_iK_n}{K_i\gamma_n + K_n\gamma_i}\frac{dp_c}{ds_i}\frac{\partial s_i}{\partial x}\right] = 0 \qquad \text{.... (2.8)}$$

A set of appropriate boundary conditions may be chosen as

$$S_i(0,t) = S_{j0}\,,\ S_i(\infty,t) = 0 \qquad \text{.... (2.9)}$$

Where the first condition defines that S_{io} is the imbibition face saturation and the second condition states the fact that the saturation at infinity remains zero. Equation (2.9) constitutes the desired differential system.

2. MATHEMATICAL SOLUTION

Since all the parameters are function of the saturation, therefore, we can write eqn. (2.8) into the form

$$p \frac{\partial s_i}{\partial t} + \frac{\partial}{\partial x}\left(K\, D(S_i) \frac{\partial s_i}{\partial x}\right) = 0 \quad \text{.... (3.1)}$$

Where $D(s_i) = \frac{k_i K_n}{K_i \gamma_n + K_n \gamma_i} \frac{dp_c}{ds_i}$ ..... (3.2)

Equations (3.1) can be rewritten in the dimensionless form by considering

$$\xi = \frac{X}{L} \text{ and } \theta = \frac{K t}{P L^2} \quad \text{.... (3.4)}$$

Substituting this value in eqn. (3.1), we get

$$\frac{\partial s_i}{\partial \theta} + \frac{\partial}{\partial \xi}\left(D(S_i) \frac{\partial s_i}{\partial \xi}\right) = 0 \quad \text{.... (3.5)}$$

This is the nonlinear differential equation of the saturation together with boundary condition

$$S_i\,(0,\infty) = S_{i0}\ ,\ S_i\,(\infty,\theta) = 0 \quad \text{.... (3.6)}$$

For obtaining the exact solution of eqn. (3.5) subject to boundary condition (3.6) we choose a similarity variable

$$\emptyset = \frac{\xi}{\sqrt{\emptyset}} \text{ (Boltzman 1894)} \quad \text{.... (3.7)}$$

When this transformation is applied to eqn. (3.5), we obtain an ordinary differential equation of the form

$$\frac{d}{d\emptyset}\left(D(S_i) \frac{dS_i}{d\emptyset}\right) - \frac{1}{2} \emptyset \frac{dS_i}{d\emptyset} = 0 \quad \text{..... (3.8)}$$

The variation of S_i between zero and one suggest that after first integration S_i may be used as the independent variable. By integration eqn. (3.8) between the limit 0 to S_i, we get

$$D(S_i) = \frac{1}{2} \frac{d\emptyset}{dS_i} \int_0^{S_i} \emptyset ds_i \quad \text{... (3.9)}$$

Since the pressure discontinuity between two flowing phases (capillary pressure) may be regarded as purely function of displacing saturation (scheidegger 1960b, pp.54,55), therefore, it can be written

$$S_i = \delta(p_c) \quad \text{.... (3.10)}$$

Then equation (3.9) can be written in terms of capillary pressure as

$$D'(p_c) = \frac{1}{2} \frac{d\emptyset}{dp_c} \int_\infty^{\delta(p_c)} \emptyset'\,(p_c)\, dp_c \quad \text{.... (3.11)}$$

Where $\emptyset'(p_c) = \emptyset\delta'(p_c)$

$$D'(p_c) = \frac{1}{2} L\left\{\frac{d\emptyset}{dp_c} \int_\infty^{\delta(p_c)} \emptyset'(p_c)\, dp_c\right\}$$

$$= \frac{1}{2} L\left\{\frac{d\emptyset}{dp_c}\right\} L\left\{\int_\infty^{\delta(p_c)} \emptyset'(p_c) dp_c\right\}$$

$$D'(p_c) = \frac{1}{2} \emptyset' \left\{s\bar{\emptyset} - \emptyset(0)\right\} \text{ where } \emptyset(0) = 0$$

$$D'(p_c) = \frac{1}{2} \emptyset' s \emptyset$$

Then by inverse Laplace Transformation

$$\emptyset\emptyset' = \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} e^s p_c \left(\frac{2}{s} D'(p_c)\right) dp_c \quad \text{.... (3.12)}$$

Equation (3.12) gives the capillary pressure function between two flowing phases.

3. PARTICULAR CASE: SOLUTION FOR CONSTANT $D(S_i)$

We assume here that the average value of $D(s_i) = \bar{D}(s_i)$ is a constant. In this case the similarity solution can be obtained as below. The equation (3.5) can be written as

$$\frac{\partial s_i}{\partial \theta} + \bar{D} \frac{\partial^2 s_i}{\partial \xi^2} = 0 \quad \text{.... (4.1)}$$

To solve eqn. (4.1), we use Brikhoff's technique of one parameter group transformation (Scheidegger 1960). Let a group T_1 consisting of a set of transformation be defined as

$$T_1 : \bar{\xi} = a^p \xi, \bar{\theta} = a^r \theta, \text{and } \bar{s}_i = a^s S_i, \quad \text{.... (4.2)}$$

Where the parameter $a \neq 0$ and p, r, s are real numbers to be determined.

$$\frac{\partial\left(\frac{\bar{S}_1}{a^s}\right)}{\partial\left(\frac{\theta}{a^r}\right)} + \bar{D}\,\frac{\partial^2\left(\frac{\bar{S}_1}{a^s}\right)}{\partial\left(\frac{\xi}{a^p}\right)^2} = 0$$

Substituting the values from eqn. (4.2) in equation (4.1), we obtain

$$a^{r-s}\,\frac{\partial \bar{S}_1}{\partial \theta} + \bar{D}\,a^{2p-s}\,\frac{\partial^2 \bar{S}_1}{\partial \xi^2} = 0 \qquad \text{.... (4.3)}$$

Equation (4.3) is absolute conformal invariant under T_1 provided

$$2p - s = r - s \qquad \text{.... (4.4)}$$

Now, we choose to eliminate θ so that the solution of eqn. (4.4) for $r \neq 0$ is equivalent to the solution of

$$\frac{2p}{r} - \frac{s}{r} = 1 - \frac{s}{r} \qquad r \neq 0 \qquad \text{.... (4.5)}$$

Choosing an arbitrary constant A and then setting

$$A = \frac{s}{r} \qquad \text{.... (4.6)}$$

Combining eqn. (4.6) with (4.5), we get

$$\frac{p}{r} = \frac{1}{2} \qquad \text{.... (4.7)}$$

Thus the invariants of the group T_1 are given by

$$\eta = \frac{\xi}{\sqrt{\theta}},\quad F(\eta) = \frac{s_i(\xi,\theta)}{\theta^A}$$

$$s_i(\xi,\theta) = F(\eta)\,\theta^A \qquad \text{.... (4.8)}$$

And the derivative of the saturation S_i in terms of $F(\eta)$ are

$$\frac{\partial s_i}{\partial \theta} = \theta^{A-1}\left[AF(\eta) - \frac{1}{2}\,\eta\,F'(\eta)\right]$$

Again $\eta = \frac{\xi}{\sqrt{\theta}}$, $F(\eta) = \frac{S_i(\xi,\theta)}{\theta^A}$

$$S_i = F(\eta)\,\theta^A$$

$$\frac{\partial S_i}{\partial \xi} = \theta^A\,F'(\eta)\,\frac{\partial \eta}{\partial \xi}$$

$$= \theta^{A-1/2}\,F'(\eta)$$

$$\frac{\partial^2 \xi}{\partial \xi^2} = \theta^{A-1}\,F''(\eta) \qquad \text{..... (4.9)}$$

Substituting these values in eqn. (4.1) we get

$$\theta^{A-1}\left[\left(AF - \frac{1}{2}\,\eta\,F'\right) + \bar{D}F''\right] = 0 \qquad \text{..... (4.10)}$$

Since $\theta^{A-1} \neq 0$ therefore, we have

$$F'' + \alpha\eta\,F' + bF = 0 \qquad \text{..... (4.11)}$$

Where $\alpha = -\frac{1}{2D}$ and $b = A/\bar{D}$

This is linear ordinary differential equation of second order whose solution can be by the following substitution.

$$F(\eta) = u(Z)\,;\quad 2Z = -\alpha\eta^2 \qquad \text{..... (4.12)}$$

$$F'(\eta) = u'(Z)\,(-\alpha\eta)$$

Put all the values in eqn. (4.11), we get

$$[\alpha^2\eta^2\,u''(Z) - u'(Z)\alpha] + \alpha\eta\,[(-\alpha\eta)u'(Z)] + b[u(Z)] = 0$$

$$Z\,u''(Z) + \left[\frac{1}{2} - Z\right]u'(Z) - \frac{b}{2\alpha}\,u(Z) = 0 \qquad \text{..... (4.13)}$$

This is called the confluent hypergeometric differential equation (Murphy 1969), whose general solution is given by

$$u(Z) = D_1 u_1 + D_2 u_2 \qquad \text{..... (4.14)}$$

Where D_1 and D_2 are constants and

$$u_1 = {}_1F_1\left(-A, \frac{1}{2}; Z\right);$$

And $u_2 = {}_1F_1\left(\frac{1}{2} - A, \frac{3}{2}; Z\right)$

The values of ${}_1F_1\left(-A, \frac{1}{2}; Z\right)$ and $\sqrt{Z}\;{}_1F_1\left(\frac{1}{2} - A, \frac{3}{2}; Z\right)$

$${}_1F_1\left(-A, \frac{1}{2}; Z\right) = \sum_{k=0}^{\infty} B_k Z^k$$

Using hyper geometric series

$${}_1F_1\left(-A, \frac{1}{2}; Z\right) = 1 + \frac{aZ}{b} + \frac{a(a+1)}{b(b+1)}\,\frac{Z^2}{2!} + \cdots$$

Here a= A and $b = \frac{1}{2}$

$$1 - \frac{AZ}{\frac{1}{2}} + \frac{(-A)(-A+1)}{\frac{1}{2}\left(\frac{1}{2}+1\right)}\frac{Z^2}{2!} + \cdots = B_0 + B_1 Z + B_2 Z^2 + \cdots$$

Comparing the equations on both sides, we get

$$B_k = \frac{-A\,(1-A)\,\ldots\,(k-A)}{\frac{1}{2}\cdot\frac{3}{2}\cdots\left(k+\frac{1}{2}\right)k!}$$

$$\sqrt{Z}\ {}_1F_1\left(\frac{1}{2}-A,\ \frac{3}{2};Z\right) = \sum_{k=0}^{\infty} C_k Z^k$$

Here $a = \frac{1}{2} - A$ and $b = \frac{3}{2}$

$${}_1F_1\left(\frac{1}{2}-A,\ \frac{3}{2};Z\right) = 1 + \frac{aZ}{b} + \frac{a(a+1)}{b(b+1)}\frac{Z^2}{2!} + \cdots$$

$$1 + \frac{\left(\frac{1}{2}-A\right)Z}{\frac{3}{2}} + \frac{\left(\frac{1}{2}-A\right)\left(\frac{3}{2}-A\right)}{\frac{3}{2}\cdot\frac{5}{2}}\frac{Z^2}{2!} + \cdots = C_0 + C_1 Z + C_2 Z^2 + \cdots$$

Comparing the equations on both sides, we get

$$C_k = \frac{\left(\frac{1}{2}-A\right)\left(\frac{3}{2}-A\right)\ldots\left(k-A+\frac{1}{2}\right)}{\frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{3}{2}+k\right)k!}$$

Where A is free parameter and ${}_1F_1\,(\beta,\delta;Z)$ denotes the confluent hyper geometric function of argument Z and parameter β and δ. Then the solution of equation (4.13) in terms of hypergeometric function is

$$u(Z) = C_1\ {}_1F_1\left(-A,\frac{1}{2};Z\right) + C_2\sqrt{Z}\,{}_1F_1\left(\frac{1}{2}-A,\frac{3}{2};Z\right) \ldots (4.15)$$

Equation (4.15) gives a formal analytic solution of the nonlinear differential equation for finger-imbibition under the specifications of the investigated problem. The constant C_1 and C_2 may be evaluated from particular specification for the problem.

3. CONCLUSIONS

In this paper we have obtained the analytical solution of the nonlinear differential equation of porous media by using a similarity technique and the possibility for deriving an expression for the wetting phase saturation in exact form has been discussed.Notwithstanding the limitation of the present analysis it is believed that the similarity solution for a complicated flow problem in porous media which is obtained here will have relevance to some physical problem useful in analytical study.

REFERENCES

1.Ames, W.F.; nonlinear Ordinary differential equation in transport process, Acad press, New York, 1968, P. 128.

2. Birkhoff, G.(1960). Hydrodynamics Princeton University Press, Princeton, New Jersey, (chapter 4 and 5).

3.Graham J.W., and Richardson, J.G.(1959). Theory and application of imbibition phenomena in recovery of oil. Trans. A.I.M.E., 216,377.

4.Murphy, G.M. (1969). Ordinary Differential Equation and Their solutions. East- West press, New Delhi, pp.323,331

5.Rijik, V.M. (1960). O Kapilliarnoi Proptke Vodoineft Cenasisennogo ghidrofilnogo plasta. Izv. Akad. Nauk SSSR, otd . Tekhn. Nauk, Mekhan. I Mashinostr, 2.

6. S.S.Patel, An exact solution of imbibition phenomena in two phase flow through porous media, The Mathematical Education, Vol.XXXTV , No.1, March 2000

7.Scheidegger, A.E. (1960a). The Physics of flow through Porous Media. University of Toronto press, pp.201, 216, 229.

8. Verma.A.P. (1968). Motion of immiscible liquids in a cracked heterogeneous porous medium with capillary pressure. Rev. Roumaine Sci. Tech. Mech.Appl.,13,2,277-92.