A Solution of Fluid flow through Porous medium Equation by Homotopy Perturbation Transform Method

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Abstract - *The present Paper discusses analytically the phenomenon of imbibition in two immiscible fluid flow through porous media. In this paper, we apply the Homotopy perturbation transform method (HPTM) using He's Polynomial for finding the analytical solution of porous medium equation. The HPTM method is a combination of Laplace Transform method and the Homotopy perturbation method. This method is very efficient, simple and can be applied for other non - linear problems also.*

***Key Words*: Homotopy perturbation method, Laplace Transform method, He's Polynomial, Porous medium equation.**

1. INTRODUCTION

It is well known that when a porous medium of Length L, field with some fluid (N) is brought into contact with another fluid (I) which preferentially wets the medium, it is observed that there is a spontaneous flow of the wetting fluid into the medium and a counter flow of the resident fluid flow the medium. This phenomena is called imbibition and has been discussed by many authors from different viewpoints, e.g. Mehta[2],Verma[7],Scheidegger[4]

The nonlinear diffusion equation is a prominent example of porous medium equation. Most phenomena describe by nonlinear equations are still difficult to obtain accurate results and often more difficult to get an analytic approximation than a numerical one. The results are obtained by many techniques as Adomian's decomposition methods, The Variational iteration method, The Laplace decomposition are divergent in most cases and which results in causing a lot of chaos. These methods have their own limitation like the calculation of Adomain's polynomial and the Langrange's multipliers.

1.1 HOMOTOPY PERTURBATION TRASFORM METHOD(HPTM)

This method has been introduced by Y. Khan and Q. Wu [1] by Combining the Homotopy Perturbation method and Laplace Transform method for solving various and non-linear systems of partial differential equations.

To illustrate the basic idea for HTPM, we consider a general nonlinear partial differential with the initial conditions of the form.

$$D_u(x,t) + R_u(x,t) + N_u(x,t) = g(x,t) \qquad (1)$$

$$\mathrm{u}(x,t) = h(u) \qquad \& \qquad u_t(x,0) = f(x)$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less than D, N represent the general non-linear differential operator and $g(x,t)$ is the source term.

Taking the Laplace Transform (denoted in this paper by L) on both the sides of eq. (1):

$$L[D\,u(x,t)] + L[R\,u(x,t)] + L[N\,u(x,t)] = L[g(x,t)] \qquad (2)$$

Using the differential property of the Laplace Transform, We have

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s} - \frac{1}{s^2}L[R\,u(x,t)] - \frac{1}{s^2}L[N\,u(x,t)] + \frac{1}{s^2}L[g(x,t)] \qquad (3)$$

Operating with the Laplace inverse on both the Sides of eq. (3) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2} L[R\,u(x,t) + N\,u(x,t)]\right] \quad (4)$$

Where $G(x,t)$ represents the term arising from source term and the prescribed initial condition.

Now we apply the Homotopy Perturbation method,

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (5)$$

and the nonlinear term can be decomposed as

$$N_u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (6)$$

For some He's polynomials $H_n(u)$ that are given by

$$H_u(u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0,1,2,3 \dots \quad (7)$$

Substituting eq. (7),eq.(6) and eq.(5) in eq.(4) we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2} L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \quad (8)$$

Which is the coupling of the Laplace Transform and the Homotopy Perturbation method using He's polynomials, Comparing the coefficient of like powers of p, the following approximations are obtained.

$$p^0 : \; u_0(x,t) = G(x,t)$$

$$p^1 : \; u_1(x,t) = -\frac{1}{s^2} L[R\,u_0(x,t) + H_0(u)]$$

$$p^2 : \; u_2(x,t) = -\frac{1}{s^2} L[R\,u_1(x,t) + H_1(u)]$$

$$p^3 : \; u_3(x,t) = -\frac{1}{s^2} L[R\,u_2(x,t) + H_2(u)]$$

………… (9)

The best approximation for the solutions is

$$u = \lim_{u=1} u_n = u_0 + u_1 + u_2 + \cdots \quad (10)$$

1.2 Resolution and results

PROBLEM REVIEW

Discuss analytically the phenomenon of imbibition in two immiscible fluid flow through porous medium and get solution HPTM for porous medium equation.

PROBLEM: I

The Non- linear diffusion equation of porous medium equation is [3],

$$\frac{\partial s_i}{\partial t} = \frac{\partial}{\partial x}\left[S_i \frac{\partial s_i}{\partial x}\right] \quad (11)$$

With initial condition $S_i(x,0) = e^x$

Applying Laplace Transform on both the side of equation (11), subject to the initial condition

$$L\left[\frac{\partial s_i}{\partial t}\right] = L\left[\left(\frac{\partial si}{\partial x}\right)^2\right] + L\left[S_i \frac{\partial^2 (S_i)}{\partial x^2}\right] \quad (12)$$

This can be written on applying the above specified initial condition as,

$$S_i(x,t) = \frac{1}{s}(e^x) + \frac{1}{s} L\left[\left(\frac{\partial s_i}{\partial x}\right)^2\right] + \frac{1}{s} L\left[S_i \frac{\partial^2 (S_i)}{\partial x^2}\right] \quad (13)$$

Taking inverse Laplace on both sides, we get,

$$L^{-1}\left[S_i(x,s)\right] = e^x L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left|\frac{1}{s}\ L\left[\left(\frac{\partial s_i}{\partial x}\right)^2 + \left(S_i\ \frac{\partial^2(S_i)}{\partial x^2}\right)\right]\right|$$

$$S_i(x,t) = e^x + L^{-1}\left|\frac{1}{s}\ L\left[\left(\frac{\partial s_i}{\partial x}\right)^2 + \left(S_i\ \frac{\partial^2(S_i)}{\partial x^2}\right)\right]\right| \quad (14)$$

Now applying the homotopy perturbation method in the form,

$$S_i(x,t) = \sum_{n=0}^{\infty} p^n\left(u_n(x,t)\right) \quad (15)$$

Eq. (5) can be reduces to,

$$\sum_{n=0}^{\infty} p^n\left(u_n(x,t)\right) = e^x + L^{-1}\left|\frac{1}{s}\ L\left[\left(\left(\sum_{n=0}^{\infty} p^n(u_n(x,t))\right)_x\right)^2 + \left(\sum_{n=0}^{\infty} p^n(u_n(x,t))\right)\left(\sum_{n=0}^{\infty} p^n(u_n(x,t))\right)_{xx}\right]\right| \quad (16)$$

On expansion of equation (16) and Comparing the coefficient of various powers of p,

We get

$$p^0 : \ u_0(x,t) = e^x$$

$$p^1 : \ u_1(x,t) = L^{-1}\left[\frac{1}{s}\ L\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right]\right] + L^{-1}\left[\frac{1}{s}\ L\left[u_0\ \frac{\partial^2(u_0)}{\partial x^2}\right]\right]$$

$$p^2 : u_2(x,t) = L^{-1}\left[\left[\frac{1}{s}\ L\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)\right]\right]\right] + L^{-1}\left|\frac{1}{s}\ L\left[u_1\frac{\partial^2(u_0)}{\partial x^2} + u_0\frac{\partial^2(u_1)}{\partial x^2}\right]\right|$$

……………………… (17)

Proceeding in similar manner we can obtain further values, Substituting above values in equation we get solution in the form of a series,

$$u(x,t) = u_0 + u_1 + u_2 + \cdots \ldots\ldots\ldots$$

Proceeding in similar manner we can obtain further values, Substituting above values in equation we get solution in the form of a series,

$$u(x,t) = u_0 + u_1 + u_2 + \cdots \ldots\ldots\ldots$$

$$\mathbf{u(x,t) = e^x + 2\,e^{2x}\,t + 9t^2 e^{3x} + \cdots \ldots\ldots}$$

which is the required solution.

PROBLEM: (II)

The Non- linear diffusion equation of porous medium equation is,

$$\frac{\partial s_i}{\partial t} = \frac{\partial}{\partial x}\left[S_i\ \frac{\partial s_i}{\partial x}\right] \quad (18)$$

With Initial condition $S_i(x,0) = -x$

Applying Laplace Transform on both the side of equation (18), subject to the initial condition

$$\mathbf{L}\left[\frac{\partial s_i}{\partial t}\right] = L\left[\left(\frac{\partial s_i}{\partial x}\right)^2\right] + L\left[S_i\ \frac{\partial^2(S_i)}{\partial x^2}\right] \quad (19)$$

This can be written on applying the above specified initial condition as,

$$S_i(x,t) = \frac{1}{s}(-x) + \frac{1}{s}L\left[\left(\frac{\partial s_i}{\partial x}\right)^2\right] + \frac{1}{s}\ L\left[S_i\ \frac{\partial^2(S_i)}{\partial x^2}\right] \quad (20)$$

Taking inverse Laplace on both sides, we get,

$$L^{-1}\left[S_i(x,s)\right] = (-x)\,L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left|\frac{1}{s}\ L\left[\left(\frac{\partial s_i}{\partial x}\right)^2 + \left(si\ \frac{\partial^2(S_i)}{\partial x^2}\right)\right]\right| \quad (21)$$

$$S_i(x,t) = (-x) + L^{-1}\left|\frac{1}{s}\ L\left[\left(\frac{\partial s_i}{\partial x}\right)^2 + \left(S_i\ \frac{\partial^2(S_i)}{\partial x^2}\right)\right]\right|$$

Now applying the homotopy perturbation method in the form,

$$S_i(x,t) = \sum_{n=0}^{\infty} p^n (u_n(x,t)) \quad (22)$$

Eq. (5) can be reduces to,

$$\sum_{n=0}^{\infty} p^n (u_n(x,t)) = (-x) + L^{-1}\left[\frac{1}{s} L\left[\left(\left(\sum_{n=0}^{\infty} p^n(u_n(x,t))\right)_x\right)^2 + \left(\sum_{n=0}^{\infty} p^n(u_n(x,t))\right)\left(\sum_{n=0}^{\infty} p^n(u_n(x,t))\right)_{xx}\right]\right] \quad (23)$$

On expansion of equation (23) and comparing the coefficient of various powers of p,

We get,

$$p^0 : \quad u_0(x,t) = (-x)$$

$$p^1 : \quad u_1(x,t) = L^{-1}\left[\frac{1}{s} L\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right]\right] + L^{-1}\left[\frac{1}{s} L\left[u_0 \frac{\partial^2(u_0)}{\partial x^2}\right]\right]$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s} L\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)\right]\right] + L^{-1}\left[\frac{1}{s} L\left[u_1 \frac{\partial^2(u_0)}{\partial x^2} + u_0 \frac{\partial^2(u_1)}{\partial x^2}\right]\right]$$

.... (24)

In this case the values obtained as $u_0 = -x$, $u_1 = t$ and $u_2 = 0$ which follows

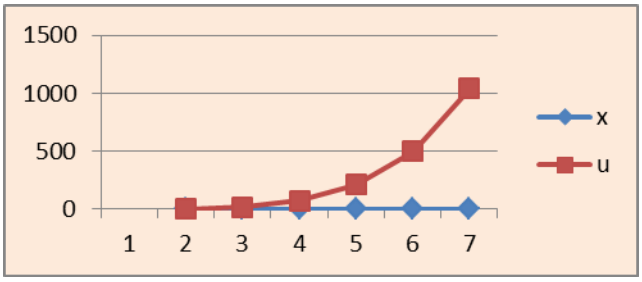
$$u_n(x,t) = 0 \; for \; n \geq 2 .$$

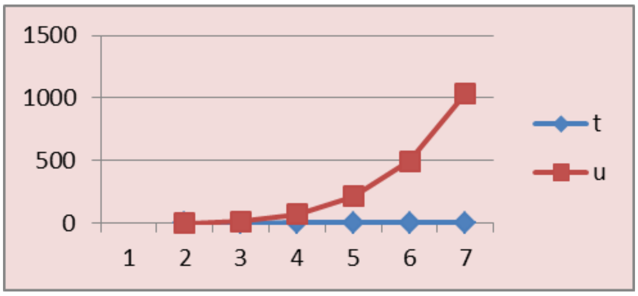
Putting these values in (11) we get the solution as

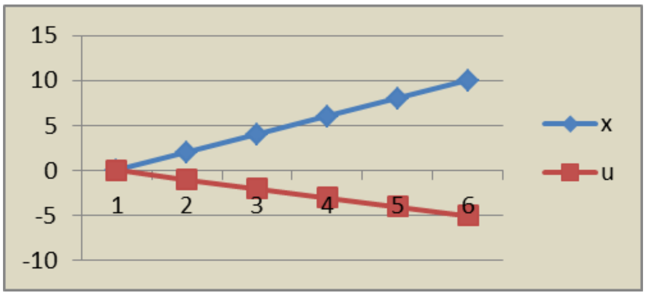
$$\boldsymbol{u(x,t) = x + t}$$

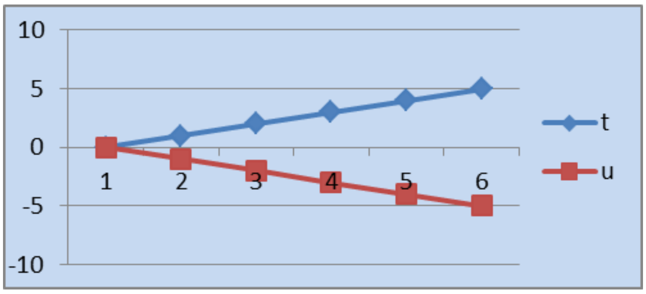
This is same as the exact solution given in [5]

2) GRAPHS:









3) CONCLUSION:

In this paper, the solutions for fluid flow through porous medium or nonlinear diffusion equation obtained with different initial conditions by homotopy perturbation transform method. From this method we can conclude that the nonlinear problems have the desired solution. From the graphs we can conclude that the saturation is in the form of exponential form in the first case and in linear form in the second case.

4) REFERENCES:

1 Khan Y, Wu Q, Homotopy perturbation transform method for nonlinear equation using He's polynomial, Journal of Computers and Mathematics with Application,Elsevier,2010

2 Mehta, M.N : J. Indian Acad.Maths,Vol.16, No.2, 1994, P.157

3 S.S. Patel, An exact solution of imbibition phenomena in two phase flow through porous media, The Mathematical Education, Vol.XXXTV , No.1, March 2000

4 Scheidegger, A.E. :University of Toronto press, 1960,P.54,61,229,231

5 Sari M, Solution of the porous media equation by a compact finite differential method, Mathematical problem in Engineering, Hindawi Publishing corporation, 2009

6 Ulrich Olivier Dangui- Mbai, Liancun Zheng, Xinxin Zhang, AJER, Volume – 3, Issue-9, PP – 224-232

7 GuptaV.G., Gupta S., Homotopy Perturbation Transform method for solving initial boundary value problem of variable co-efficient, International Journal of Nonlinear Science, Vol.12(2011) No.3, PP.270-277

8 Verma, A.P. : Canadian I. of phy., 1969, P-22