Transient analysis of reliability with and without repair for K-out-of-N: G systems with three failure modes

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Abstract : This paper presents Markov models for transient analysis of reliability with and without repair for K-out-of-N:G systems subject to three failure modes. The reliability of repairable systems can be calculated as a result of the numerical solution of a simultaneous set of linear differential equations. Closed form solutions of the transient probabilities are used to obtain the reliability for non-repairable systems. The probabilities so obtained can be utilized to estimate the reliability and MTTF of the system.

Keywords: K-out-of-N:G; Mean time between failures; Reliability; Exponential distribution.

1. Introduction

One way to improve system reliability is to add redundant components. A common form of redundancy is K-out-of-N. A K-out-of-N system functions if and only if at least K components works or equivalently at most N-K components fail. Thus the life of a non-repairable K-out-of-N system can be characterized by the (*N-K+1*) th order statistic, which represents (N-K+1) th smallest failure time among n possible times. K-out-of-N systems has been investigated extensively in the literature. Various related models have been developed and many formulas have been derived, resulting in a large body of literature, for those general results, please refer to Kapur and Lamberson[1], Singh and Billinton[2], Ravichandran[3], Trivedi[4] and Kumar and Sharma[5] etc.

Many systems consist of components that may fail in three mutually exclusive ways and the system likewise may fail in either of three mutually exclusive ways. Several authors have considered a K-out-of-N systems subject to two failure modes; see Pham and Pham[6] and reference cited therein. Moustafa[7] considered Markov models for analyzing the availability of K-out-of-N systems subject to m failure modes and obtained closed form solutions of the steady state availability.

The objective of this paper is to provide a set of simultaneous linear differential equations for repairable and non-repairable events for K-out-of-N:G systems subject to three failure modes with the condition that there is no transition between three failure modes. Numerical solutions for the reliability of the repairable systems are discussed while closed form solutions of the reliability for the non-repairable systems are derived.

The paper is organized as follows- in section 2, the general assumptions for K-out-of-N:G system are given. The transient reliability with repair is discussed and the closed form solutions of the reliability with repair are given in section 3 and 4, respectively. Conclusions are given in section 5.

2 Assumptions

1. The system consists of N identical and independent components.
2. Each component is either up (good), or failed by mode-m with constant failure rate $\lambda_m, m = 1,2,3.$
3. The system is completely observable, i.e. there is perfect information to determine instantaneously the mode of failure.
4. There is a single repair facility. The time to repair mode-m failure is exponentially distributed with mean equal to $\frac{1}{\mu_m}, m = 1,2,3.$ Repair is perfect, i.e. the repaired component is as good as new.
5. The system is up (good) if at least K components are up (good).
6. The system cannot return to the up (good) state when (N-K+1) components fail.
7. There are no transitions between the three failure modes, which means they are mutually exclusive (refs 6,7).

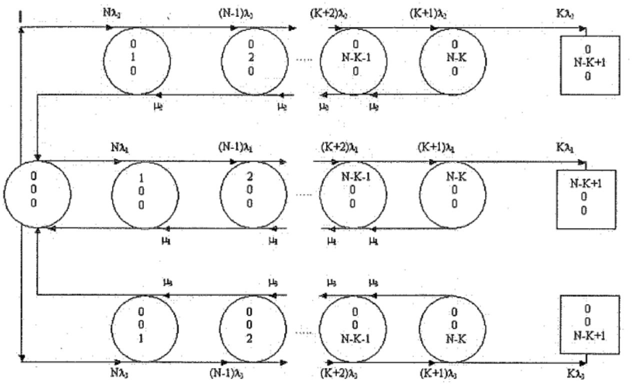

Fig. 1. State transition rate diagram

○ : Up State
□ : Down State

3 System reliability with repair

The system can be modeled by a continuous-time Markov process (refs 8). Let (*i,j,k*) be the state of the system, where i,j and k represent the number of failed components due to failure mode-1, mode-2 and mode-3 respectively, see fig. 1. Let $P_1(i,j,k)$ be the probability of being in state (*i,j,k*) at time t, when the system starts at time *t*=0 in the state (0,0,0). The set of Chapman-Kolmogorov's equations based on the transition rate diagram given in fig. 1 is.

3.1 For *i*=*j*=*k*=0

$$\frac{dP_t(0,0,0)}{dt} = -N(\lambda_1+\lambda_2+\lambda_3)P_t(0,0,0)+\mu_1 P_t(1,0,0)+\mu_2 P_t(0,1,0)+\mu_3 P_t(0,0,1) \qquad (1)$$

3.2 For *j*=*k*=0

If $1 \le i \le N-K-1$

$$\frac{dP_t(i,0,0)}{dt} = -[(N-i)\lambda_1+\mu_1]P_t(i,0,0)+[N-(i-1)]\lambda_1 P_t(i-1,0,0)+\mu_1 P_t(i+1,0,0) \qquad (2)$$

If $i = N-K$

$$\frac{dP_t(N-K,0,0)}{dt} = -[(N\lambda_1+\mu_1)P_t]P_t(N-K,0,0)+[K+1]\lambda_1 P_t(N-K-1,0,0) \qquad (3)$$

If $i = N-K+1$

$$\frac{dP(N-K+1,0,0)}{dt} = K\lambda_1 P_t(N-K,0,0) \qquad (4)$$

3.3 For $i = k = 0$

$If\, 1 \le j \le N-K-1$

$$\frac{dP_t(0,j,0)}{dt} = -[(N-j)\lambda_2+\mu_2]P_t(0,j,0)+[N-(j-1)]\lambda_2 P_t(0$$

$If\;\; j = N-K$

$$\frac{dP(0,N-K,0)}{dt} = -[K\lambda_2+\mu_2]P_t(0N-K,0,)+[K+1)]\lambda_2 P_t(0$$

$If\;\; j = N-K+1$

$$\frac{dP(0,N-K+1,0)}{dt} = K\lambda_2 P_t(0,N-K,0)$$

3.4 For $i = j = 0$

$If\;\; 1 \le k \le N-K-1$

$$\frac{dP_t(0,0,k)}{dt} = -[(N-k)\lambda_3+\mu_3]P_t(0,0,k)+[N-(k-1)]\lambda_3 P_t(0$$

$If\;\; k = N-K$

$$\frac{dP_t(0,0,N-k)}{dt} = -[(K\lambda_3+\mu_3]P_t(0,0,N-K)+[K+1)]\lambda_3 P_t(0,0$$

$If\;\; k = N-K+1$

$$\frac{dP(0,0,N-K+1)}{dt} = K\lambda_3 P_t(0,0,N-K)$$

Equations (1)-(10) form a set of first order, constant coefficient and linear differential equations with the following initial conditions-

$P_0(0,0,0) = 1 and P_0(i,j,k) = 0\; for\; all\, i,j\, and\, k > 0$

The exact numerical P_t (*i,j,k*) can be derived by determining the Eigen values of the model transition rate matrix. Dyer[9] obtained an approximate solution, which did not require the determination of Eigen values. Also, the uniformazation technique can be used for solving the given linear differential equations (refs 10)-

The transient reliability with repair R_t (with repair) and the mean time between failure (MTBF) of the system can be calculated as follows-

$$R_t(with\,repair) = \sum_{m=0}^{N-K} \sum_{\substack{(i,j,k)\\ i+j+k=m}} P_1(i,j,k) \quad (12)$$

and

$$MTBF = \int_0^{\infty} R_t(with\,repair)dt \quad (13)$$

4 System reliability without repair

Let $\mu_1 = \mu_2 = \mu_3 = 0$ and take Laplace transformations of equations (1)-(3), (5), (6), (8), and (9); the following recursive formulas can be derived -

For i=j=k=0

$$P_s(0,0,0) = \frac{1}{[s + N(\lambda_1 + \lambda_2 + \lambda_3)]} \quad (14)$$

For $j = k = 0\, and\, 1 \le i \le N-K$

$$P_s(i,0,0) = \prod_{n=1}^{i} \frac{[(N-n+1)\lambda_1]}{[s + N(\lambda_1 + \lambda_2 + \lambda_3)]} \prod_{n=1}^{i} [s + (N-n)\lambda_1] \quad (15)$$

For $i = k = 0\, and\, 1 \le k \le N-K$

$$P_s(0,j,0) = \prod_{n=1}^{j} \frac{[(N-n+1)\lambda_2]}{[s + N(\lambda_1 + \lambda_2 + \lambda_3)]} \prod_{n=1}^{k} [s + (N-n)\lambda_2] \quad (16)$$

For $i = j = 0\, and\, 1 \le k \le N-K$

$$P_s(0,0,k) = \prod_{n=1}^{k} \frac{[(N-n+1)\lambda_3]}{[s + N(\lambda_1 + \lambda_2 + \lambda_3)]} \prod_{n=1}^{k} [s + (N-n)\lambda_3] \quad (17)$$

To solve equation (15), we have used the inverse Laplace transform. The approach is to expand the left hand side fraction in the following partial fraction form, (see refs 11)-

$$\frac{1}{s + c_0} \prod_{n=1}^{i} (s + c_n) = \frac{a_0}{s + c_0} + \sum_{n=1}^{i} \frac{a_n}{s + c_n}$$

$$c_n \ne c_p\; for\, any\, n \ne p, and\, n, p = 1,2,3,\ldots,i. \quad (18)$$

Using linear algebra and mathematical induction, the result is -

$$L^{-1}\left[\frac{1}{s + c_o} \prod_{n=1}^{i} (s + c_n)\right] = \frac{e^{-c_o t}}{\prod_{n=1}^{i} (c_n - c_0)} + \sum_{n=1}^{i} \frac{e^{-c_n t}}{c_o - c_n} \prod_{\substack{p=1\\ p\ne n}}^{i} (c_p - c$$

Using equation (19), we obtain the inverse Laplace transform for equation (15)-

$$P_t(i,0,0) = M_i \left[\frac{e^{-N(\lambda 1 + \lambda 2 + \lambda 3)t}}{\prod_{n=1}^{i} \{(N-n)\lambda_1 - N(\lambda_1 + \lambda_2 + \lambda_3)\}} \right.$$

$$\left. + \sum_{n=1}^{i} \frac{e^{-[(N-n)\lambda_1]t}}{(n\lambda_1 + N\lambda_2 + N\lambda_3)} \times \prod_{\substack{p=1\\ p\ne n}}^{i} [N-p)\lambda_1 - (N-n)\lambda_1] \right] \quad (20)$$

Where $$M_i = \prod_{n=1}^{i} [N-n+1]\lambda_1 = \frac{N!\lambda_i}{(N-i)!} \quad (21)$$

and

$$\prod_{\substack{p=1\\p\neq n}}^{i}[n-p]\lambda_1=\frac{(-1)^{i-n}\binom{i}{n}^{-1} i!\lambda_1^{\,i-1}}{n}\ for n<i$$

(22)

By using (21) and (22), the equation (20) will take the following form -

$$P_t(i,0,0)=(-1)^i\left[\frac{N!}{(N-i)!}\right]\times\left[\frac{\lambda_1 e^{-N(\lambda_1+\lambda_2+\lambda_3)t}}{\prod_{n=1}^{i}(n\lambda_1+N\lambda_2+N\lambda_3)}\right]$$

$$+\binom{N}{i}\sum_{n=1}^{i}(-1)^{i-n}\binom{i}{n}\times\left[\frac{n\lambda_1}{n\lambda_1+N\lambda_2+N\lambda_3}\right]e^{-(N-n)\lambda_1 t}$$

(23)

Similarly, the inverse Laplace transformation of equation (16) and (17) are as follows -

$$P_t(0,j,0)=(-1)^j\left[\frac{N!}{(N-j)!}\right]\times\left[\frac{\lambda_j e^{-N(\lambda_1+\lambda_2+\lambda_3)t}}{\prod_{n=1}^{j}(n\lambda_1+N\lambda_2+N\lambda_3)}\right]$$

$$+\binom{N}{j}\sum_{n=1}^{i}(-1)^{j-n}\binom{j}{n}\times\left[\frac{n\lambda_2}{n\lambda_1+N\lambda_2+N\lambda_3}\right]e^{-(N-n)\lambda_2 t}$$

(24)

$$P_t(0,0,k)=(-1)^k\left[\frac{N!}{(N-k)!}\right]\times\left[\frac{\lambda_3 e^{K-N(\lambda_1+\lambda_2+\lambda_3)t}}{\prod_{n=1}^{k}(n\lambda_1+N\lambda_2+N\lambda_3)}\right]$$

$$+\binom{N}{k}\sum_{n=1}^{k}(-1)^{k-n}\binom{k}{n}\times\left[\frac{n\lambda_3}{n\lambda_1+N\lambda_2+N\lambda_3}\right]e^{-(N-n)\lambda_3 t}$$

(25)

While the inverse Laplace transforms of equation (14) is -

$$P_t(0,0,0)=e^{-N(\lambda_1+\lambda_2+\lambda_3)t}$$

(26)

Using equations (23)-(26), the transient reliability without repair R_t (without repair) and the mean time to failure (MTTF) can be calculated as follows-

$$R_t(without\,repair)=\sum_{m=0}^{N-K}\sum_{\substack{(i,j,k)\\i+j+k=m}}P_t(i,j,k)$$

(27)

and

$$MTTF=\int_0^{\infty}R_t(Without\,repair)dt$$

(28)

In particular, if $\lambda_2=\lambda_3=0, and\ \lambda_1=\lambda.$ equation (27) is reduced to-

$$R(t)=\sum_{i=0}^{N-K}\binom{N}{i}[1-e^{-\lambda t}]^i\times e^{-(N-i)\lambda t}$$

(29)

and

$$MTTF = \sum_{i=0}^{N-K} \frac{1}{\lambda(N-i)} \quad (30)$$

which agree with the reliability function and the mean time to failure for K-out-of-N:G system with single failure mode.

5 Conclusions

Markov models have been used to derive the transient reliability and the mean time between failures for repairable K-out-of-N:G systems subject to three failure modes. Closed form solutions of the transient probabilities for non-repairable systems are given. These probabilities can be used to find the reliability and the mean time to system failure of the systems.

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