On the generalization of Fourier series, Fourier-type integral transform and extending the idea to a generalized double series and transform

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Abstract - *In this paper we introduce a new generalized Fourier series and a Fourier-type integral transform. We derive the complex inversion formula, convolution theorem and generalized product theorem. We also attempt to generalize the convolution of two functions and the Dirac-delta function. In the last section we extend our idea to a generalized double Fourier series and transform which can be further extended to n-Fourier series and transform.*

Key Words: Fourier series, Fourier transform, convolution, Dirac-delta function

1.INTRODUCTION

Many linear BVPs and IVPs in applied mathematics, mathematical physics and engineering science can be effectively solved by the use of Fourier transform. This transform is very useful for solving differential and integral equations for the following reasons. Firstly, these equations are replaced by simple algebraic equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Secondly, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Thirdly, the transform solution combined with the convolution theorem provides an elegant representation of the solution of the BVPs and IVPs. Methods based on the Fourier transform are used in virtually all areas of engineering and science. We expect that many fields and many interests will be represented in such fields, and this brings up an important issue for all of us to be aware of. With the diversity of interests and backgrounds present not all examples and applications will be familiar and of relevance to all people. Sometimes it becomes important to not just think of application but to concentrate upon the beauty of the theory of the subject. We'll all have to cut each other some slack. We took our chance in this paper to just analyze and discuss the theoretical aspects and results. Along the same lines, it is also important for us to accept that this is one study on the generalization among many possible courses. The richness of the subject, both mathematically and in the range of applications, means that we'll be making choices almost constantly.

2.The Generalized Fourier Series And Transform

We define a generalized Fourier series of a function f(x),with respect to ε(x),

$$\sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{in\pi\varepsilon(x)}{a}\right) \tag{2.1}$$

where the coefficients are

$$a_n = \frac{1}{2a}\int_{-a}^{a} \exp\left(-\frac{in\pi\varepsilon(t)}{a}\right)\varepsilon'(t)\, f(t)dt \tag{2.2}$$

and the range of ε(x) is equipotent with the set of real numbers.

The above representation is evidently periodic with period $2a$ in the interval $(-a,a)$ and it can represent f(x) in $(-a,a)$. However it cannot represent f(x) outside the interval $(-a,a)$ unless f(x) is periodic of period $2a$.

Here the problems on finite intervals lead to Fourier series and the problems on whole line (-∞,∞) will lead to Fourier integrals.

We now attempt to find an integral representation of a non periodic function f(x) in (-∞,∞) by letting $a \to \infty$. As the interval grows ($a \to \infty$),the values $k_n = \frac{n\pi}{a}$ become closer together and form a dense set. If we take $\Delta_k = k_{n+1} - k_n = \frac{\pi}{a}$ and substitute the coefficients a_n into equation (2.1), we obtain,

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\Delta_k\left[\int_{-a}^{a}\exp(-ik_n\varepsilon(t))\varepsilon'(t)f(t)dt\right]\exp(ik_n\varepsilon(x)) \quad (2.3)$$

In the limit as $a\to\infty$, k_n becomes a continuous variable *k*, and Δ_k becomes *dk*. Consequently, the sum can be replaced by an integral in the limit and equation (2.3) reduces to,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\exp(-ik\varepsilon(t))\varepsilon'(t)f(t)dt\right]\exp(ik\varepsilon(x))dk \quad (2.4)$$

We consider the Fourier type integral transform,

$$F_\varepsilon\{f(x);k\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\phi(k)\varepsilon(x))\varepsilon'(x)f(x)dx=F(k) \quad (2.5)$$

where the factor $\frac{1}{\sqrt{2\pi}}$ is obtained by splitting the factor $\frac{1}{2\pi}$ involved in equation (2.4).

By this definition (2.5) it is obvious that this transform will be a generalization to the complex Fourier transform by considering $\phi(k)=k$ and $\varepsilon(x)=x$.

3.A Generalized Convolution Of Two Functions

For two functions $f(x)$,$g(x)$ and a function $\varepsilon(x)$,where the range of $\varepsilon(x)$ is equipotent with the set of real numbers, we define a generalized convolution of $f(x)$ and $g(x)$ with respect to $\varepsilon(x)$ as

$$f *_\varepsilon g=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varepsilon'(x)f(x)g\left(\varepsilon^{-1}(\varepsilon(y)-\varepsilon(x))\right)dx$$

4.Some Properties Of The Generalized Fourier Transform

In this section we establish theorems on the generalized Fourier transform. First, we derive a complex inversion formula for this transform.

Theorem 4.1 (The complex inversion formula for the generalized Fourier transform)

Let $F(\phi^{-1}(k))$ be an analytic function of *k* (assuming that $F(\phi^{-1}(k))$ has no branch point). If $F(\phi^{-1}(k))\to 0$ as $k\to\infty$ through the left plane Re(*k*) $\le c$, and

$$F_\varepsilon\{f(x);k\}=F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\phi(k)\varepsilon(x))\varepsilon'(x)f(x)dx$$

then,

$$F_\varepsilon^{-1}\{F(k)\}=f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(ik\varepsilon(x))F(\phi^{-1}(k))dk \quad (2.6)$$

Proof: By the definition of the generalized Fourier transform (2.5) and letting $\phi(k)=p$, we have

$$F(\phi^{-1}(p))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ip\varepsilon(x))\varepsilon'(x)f(x)dx\,.$$

Now, by setting $t=\varepsilon(x)$ in the above relation we obtain

$$F(\phi^{-1}(p))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ipt)f(\varepsilon^{-1}(t))dt=F\{f(\varepsilon^{-1}(t));p\}$$

At this point, by the complex inversion formula for the Fourier transform and setting back $\varepsilon^{-1}(t)=x$, $p=k$, we finally get,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(ik\varepsilon(x))F(\phi^{-1}(k))dk$$

Theorem 4.2 (The generalized Fourier transform of $_\varepsilon\delta_x$ - derivatives)

Let $f, f', f'',\ldots, f^{(n-1)}$ be continuously differentiable functions with $f^{(n)}$ being piecewise continuous in the interval [0,∞),and if $\lim_{x\to\pm\infty} f^{(r)}(x)=0$ for each $r=0,1,2,\ldots,n-1$. Then,

$$F_\varepsilon\{_\varepsilon\delta_x^n f(x);k\}=(i\phi(k))^n F(k) \quad (2.7)$$

where the $_\varepsilon\delta_x$ - derivative operator is defined as follows,

$$ {}_{\varepsilon}\delta_x \equiv \frac{1}{\varepsilon'(x)}\frac{d}{dx} $$

Proof: Using the definitions of the generalized Fourier transform (2.5) and the ${}_{\varepsilon}\delta_x$- derivative, we obtain

$$ F_{\varepsilon}\{{}_{\varepsilon}\delta_x f(x);k\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\phi(k)\varepsilon(x))\varepsilon'(x)\,{}_{\varepsilon}\delta_x f(x)\,dx $$

$$ = \frac{1}{\sqrt{2\pi}}\lim_{\substack{A\to\infty\\ B\to-\infty}}\left[\exp(-i\phi(k)\varepsilon(x))f(x)\right]_B^A $$

$$ + i\phi(k)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\phi(k)\varepsilon(x))\varepsilon'(x)f(x)\,dx $$

Since $\lim_{x\to\pm\infty} f^{(r)}(x) = 0$, it follows that

$$ F_{\varepsilon}\{{}_{\varepsilon}\delta_x f(x);k\} = (i\phi(k))F(k) $$

Consequently, by repeated application of the above relation once again, we get

$$ F_{\varepsilon}\{{}_{\varepsilon}\delta_x^2 f(x);k\} = (i\phi(k))^2 F(k) $$

and by repeating the above scheme for ${}_{\varepsilon}\delta_x^n f(x)$,we can easily arrive at (2.7)

Theorem 4.3 (The generalized convolution theorem for the generalized Fourier transform)

If F(k) and G(k) are the generalized Fourier transform of the functions $f(x)$ and $g(x)$ respectively, then

$$ F(k)G(k) = F_{\varepsilon}\{f *_{\varepsilon} g;k\} $$

and equivalently

$$ f *_{\varepsilon} g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(ik\varepsilon(x))F(\phi^{-1}(k))G(\phi^{-1}(k))dk $$

Proof: Using the definition for generalized Fourier transform for F(k) and G(k),we have

$$ F(k)G(k) = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\phi(k)\varepsilon(x))\varepsilon'(x)f(x)\,dx\right) \times\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\phi(k)\varepsilon(t))\varepsilon'(t)g(t)\,dt\right) $$

$$ = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-i\phi(k)(\varepsilon(x)+\varepsilon(t)))\,\varepsilon'(x)\varepsilon'(t)f(x)g(t)\,dx\,dt $$

Now,by substitution $\varepsilon(x)+\varepsilon(t)=\varepsilon(y)$ we get

$$ F(k)G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\phi(k)\varepsilon(y))\varepsilon'(y)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varepsilon'(x)f(x)g\left(\varepsilon^{-1}(\varepsilon(y)-\varepsilon(x))\right)dx\right]dy $$

$$ = F_{\varepsilon}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varepsilon'(x)f(x)g\left(\varepsilon^{-1}(\varepsilon(y)-\varepsilon(x))\right)dx;k\right\} = F_{\varepsilon}\{f *_{\varepsilon} g;k\} $$

From this we have equivalently,

$$ f *_{\varepsilon} g = F_{\varepsilon}^{-1}\{F(k)G(k)\} $$

$$ = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(ik\varepsilon(x))F(\phi^{-1}(k))G(\phi^{-1}(k))dk $$

5.The Generalized Dirac-Delta Function

From the definition of generalized Fourier and generalized inverse Fourier transform we get

$$ f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(ik\varepsilon(x))\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ik\varepsilon(y))\varepsilon'(y)f(y)\,dy\right)dk $$

$$ = \int_{-\infty}^{\infty}\delta(\varepsilon(x)-\varepsilon(y))f(y)\,dy, $$

where we consider $\delta(\varepsilon(x)-\varepsilon(y))$ as the generalized Dirac-delta function, and we define it as

$$ \delta(\varepsilon(x)-\varepsilon(y)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ik(\varepsilon(x)-\varepsilon(y)))\varepsilon'(y)\,dk $$

6. The Generalized Double Fourier Series And Transform

With a very much similar approach a discussed in section 2 we extend our discussion to a generalized double Fourier series.

We define the generalized double Fourier series of a two variable function f(x,y) in a square $[l_1, l_1] \times [l_2, l_2]$, with respect to a single variable function ε as,

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn} \exp\left(\frac{im\pi\varepsilon(x)}{l_1}\right) \exp\left(\frac{in\pi\varepsilon(y)}{l_2}\right) \qquad (6.1)$$

where the coefficients are

$$a_{mn} = \frac{1}{4l_1 l_2} \int_{-l_1}^{l_1} \int_{-l_2}^{l_2} \exp\left(-\frac{im\pi\varepsilon(t)}{l_1}\right) \varepsilon'(t) \exp\left(-\frac{in\pi\varepsilon(s)}{l_2}\right) \varepsilon'(t)\varepsilon'(s) f(t,s) dt ds \qquad (6.2)$$

We consider the sequences $\left\{k_m = \frac{m\pi}{l_1}\right\}_m$ and $\left\{p_n = \frac{n\pi}{l_2}\right\}_n$. As $l_1 \to \infty$ and $l_2 \to \infty$, the points of the considered sequences become closer together and form a dense set.If we take $\Delta_k = k_{m+1} - k_m = \frac{\pi}{l_1}$ and $\Delta_p = p_{n+1} - p_n = \frac{\pi}{l_2}$ and then substitute the coefficients a_{mn} into equation (6.1), we obtain,

$$\frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Delta_k \Delta_p \left[\int_{-l_1}^{l_1} \int_{-l_2}^{l_2} \exp(-ik_m\varepsilon(t)) \exp(-ip_n\varepsilon(s)) \varepsilon'(t)\varepsilon'(s) f(t,s) dt ds \right] \times [\exp(ik_m\varepsilon(x)) \exp(ip_n\varepsilon(y))] \qquad (6.3)$$

Considering the limit $l_1 \to \infty$ and $l_2 \to \infty$, k_n and p_n become a continuous variables k and p respectively, and hence equation (6.3) reduces to,

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-ik\varepsilon(t)) \exp(-ip\varepsilon(s)) \varepsilon'(t)\varepsilon'(s) f(t,s) dt ds \right] \times [\exp(ik\varepsilon(x)) \exp(ip\varepsilon(y))] dk dp \qquad (6.4)$$

From equation (6.4) we extent our idea for the expression of the generalized double integral transform and we define it as given below.

$$F_\varepsilon\{f(x,y); k, p\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-i\phi(k)\varepsilon(x)) \exp(-i\phi(p)\varepsilon(y)) \varepsilon'(x)\varepsilon'(y) f(x,y) dx dy = F(k,p) \qquad (6.5)$$

where the factor $\frac{1}{2\pi}$ is obtained by splitting the factor $\frac{1}{4\pi^2}$ involved in equation (6.4).

7. Conclusions

This was an introductory treatment of a generalized Fourier series and a Fourier-type integral transform. The objective of this paper was to provide some new results and insights in the area of the new integral transform (F_ε-transform) and to extend the idea from one dimension to two dimensions. This paper also introduces the concept of certain generalized definitions such as generalized convolution and the generalized Dirac-delta function.

References

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