FIXED POINT THEOREM OF COMPATIBLE OF TYPE (R) USING IMPLICIT RELATION IN FUZZY METRIC SPACE

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Abstract

In this paper authors prove a Common fixed print theorem for compatible mapping of type (R) in Fuzzy metric space by using implicit relation. Our result modifies as well as generalize the results of M.Koireng et al [10]. In [1] Cho et al introduced the concept of semi compatibility in the abstract D-metric space. Recently B. Singh et al [15] introduced the concept of semi compatible mapping in the context of a fuzzy metric space. The earliest important result of compatible mapping was obtained by jungck [7]. Pathak, Chang and Cho [11] introduced the concept of compatible mapping of type (P). In 2004, R.Singh et al [12] introduced the concept by idea of compatible mapping of type (R) by combing the definition of compatible mapping and compatible mapping of type (P).

Our aspire in this paper is to define compatible of type R in fuzzy metric spaces and prove some common fixed point theorem of compatible map of type (R) by generalized some interesting result [9].

Key Words: 54H25, 54E50.

1. INTRODUCTION

The concept of fuzzy sets was introduced at first by Zadeh [17] which laid the foundation of fuzzy mathematics. George and Veeramani In [5] modified the notion of fuzzy metric space introduced by Kramosil and Michalek [8]. They also obtained that every metric space induces a fuzzy metric spaces. Sessa [16] proved a generalization of commutatively which called weak commutatively. Further Jungek [7] more generalized commutatively which is called compatibility in metric space.

Sessa introduced the concept of weakly commuting mappings and Jungck defined the notions of compatible mappings in order to generalize the concept of weak commutativity and showed that weak commuting mappings are compatible but the converse is not true. In recent years, a number of a fixed point theorems and coincidence theorems have been obtained by various authors utilizing this notion. Jungck further weakened the notion of weak compatibility and Jungck and Rhoades further extended weak compatibility.

2. PRELIMINARIES AND DEFINATIONS:

Definition 2.1 [5] The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ which satisfying the following conditions:

(1) $M(x,y,t) > 0$

(2) $M(x,y,t) = 1$ If and only if $x = y$

(3) $M(x,y,t) = M(y,x,t)$

(4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$

(5) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous, for all $x,y,z \in X$ and $t,s > 0$

Example 2.1.1 : Let (X,d) be a metric space and $a*b = ab$ or $a*b = \min\{a,b\}$.

Let $M(x,y,t) = \dfrac{t}{t+d(x,y)}$ for all $x,y \in X$ and $t > 0$ then $(X,M,*)$ is a fuzzy metric space.

Definition 2.2 [13] A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ is said to be a Cauchy sequence if and only if for each $\varepsilon > 0, t > 0$ there exists $x \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$, For all $n,m \ge x_0$

The sequence $\{x_n\}$ is said to converge to a point x in X if for each $\varepsilon > 0$, $t > 0$ there exists $x_0 \in N$ such that

$$M(x_n, x, t) > 1-\varepsilon \text{ for all } n \ge x_0$$

A fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.3 [15] A pair of self mappings (A, S) of fuzzy metric space $(X,M,*)$ is said to be compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) \to 1 \quad \forall\, t > 0$$

Whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = x$, for some $x \in X$

Definition 2.4 [14] A pair (A, S) of self mappings of a fuzzy metric space is said to be semi compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} Sx_n = x$

So (A, S) is semi compatible and $Ay = Sy$ implies $ASy = SAy$ by taking $\{x_n\} = y$ and $x = Ay = Sy$.

Proposition 2.1 [2] In a fuzzy metric space $(X,M,*)$ limit of a sequence is unique.

Proof: Let $\{x_n\} \to x$ and $\{x_n\} \to y$ then

$\lim_{n\to\infty} M(x_n, x, t) = 1 = \lim_{n\to\infty} M(x_n, y, t)$

Now $M(x,y,t) \geq M(x,x_n,t/2) * M(y,x_n,t/2)$

$M(x,y,t) \geq 1*1$ (By taking limit $n \to \infty$)

$M(x,y,t) = 1$ for all $t > 0$

Thus $x = y$ and hence the limit is unique.

Proposition 2.2 [15] If (A, S) is a semi-compatible pair of self maps of a fuzzy metric space $(X,M,*)$ and S in continuous then (A,S) is compatible.

Definition 2.5 [9] Self mappings T and *S* of a fuzzy metric space *(X, M, ∗)* is said to be compatible of type (R) if

$\lim_{n\to\infty} M(STx_n, Tx_n, t) = 1$ For all t > 0

$\lim_{n\to\infty} M(SSx_n, TTx_n, t) = 1$ For all t > 0

Whenever { x_n } is a sequence in *X* such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ For some z ∈ *X*.

Lemma 2.1 [15] let (X, M, *) be a fuzzy metric space
If there exists k ϵ X such that
M(x, y, k t) ≥ M(x, y, t/k^n) for positive integer n
taking limit as n → ∞,
M(x, y, k/t) ≥ 1 and hence x = y.

Lemma 2.2 [14] The only t-norm * satisfying r*r ≥ r for all r∈[0,1] is the minimum t-norm, that is,
a* b = min {a , b} for all a, b ∈ [0,1]

Lemma 2.3 [13] :- Let (X, M, *) be a fuzzy metric space, if there exist k (0, 1) such that M (x, y, kt) M (x, y, t) for all x, y x and t>0, then x = y.

Proposition 2.3 [9] Let (X, M, *) be a fuzzy metric space and let T and S be Compatible mappings of type(R) and T z = S z for some z ∈ X then TTz = TSz = STz = SSz.

Proposition 2.4 [9] Let (X , M , *) be a fuzzy metric space and let A and S be Compatible mappings of type(R) and let T x_n , S x_n → z as n →∞ for some z∈ X.
Then

(I) $\lim_{n\to\infty} M(STx_n, Tz, z) = 1$ for if S is continuous at t

(ii) $\lim_{n\to\infty} M(STx_n, Sz, z) = 1$ for if T is continuous at t

(iii) TSz = STz and Tz=Sz if T and S are continuous at z.

A Class of Implicit Relation –

Let ϕ be the set of all real and continuous from $\phi : [0,1]^5 \to R$ satisfying the following conditions:

(A-1) ϕ Is non-increasing in second, third, fourth and fifth argument

(A-2) $\phi(u,v,u,v,v) \geq 0 \;\Rightarrow\; u \geq v$

$\phi(u,v,v,v,v) \geq 0 \;\Rightarrow\; u \geq v$

Example:- $\phi(t_1,t_2,t_3,t_4,t_5) = t_1 - Max.\{t_2,t_3,t_4,t_5\}$

3. MAIN RESULT

THEOREM 3.1

Let A, B, S and T is self mappings of a complete fuzzy metric space $(X,M,*)$ with continuous t-norm defined by $a*b = \min\{a*b\}\{a,b\} \in [0,1]$ satisfying the following conditions:

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$

(ii) One of A, B, S and T are continuous.

(iii) Pairs (A, S) and (B, T) are compatible of type (R)

(iv) $\exists$ Some $k \in (0,1)$ such that for all $x, y \in X, t > 0$

$$\phi\big(M(Ax,By,kt), M(Sx,Ty,t), M(Sx,Ax,t), M(Ty,By,kt), M(Ty,Ax,t) \big) \geq 0$$

(v) $\forall x, y \in X, M(x,y,t) \to 1$ as $t \to \infty$

Then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any point as $A(X) \subseteq T(X)$ and $S(X) \subseteq B(X)$, $\exists x_1 \in X$ and $x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively we construct a sequence $\{y_n\}$ in X such that

$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$ And

$y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$; $(y_{2n} = Sx_{2n})$ n = 0, 1

With $x = x_{2n}, y = x_{2n+1}$ using contractive condition, we get

$$\phi(M(Ax_{2n}, Bx_{2n+1}, kt), M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, kt), M(Tx_{2n-1}, Ax_{2n}, t)\ ) \geq 0$$

$$\phi(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\ M(y_{2n+1}, y_{2n+2}, kt),\ M(y_{2n+1}, y_{2n+1}, t)) \geq 0$$ And

$$\phi(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\ M(y_{2n+1}, y_{2n+2}, kt), 1) \geq 0$$

Since ϕ is non-increasing and ψ is non decreasing in fifth argument therefore,

$$\phi(M(y_{2n-1}, y_{2n-2}.kt),)M(y_{2n}, y_{2n-1}.t), M(y_{2n}, y_{2n-1}.t), M(y_{2n-1}, y_{2n-2}.kt), M(y_{2n-1}, y_{2n-1}.t)\ ) \geq 0$$

Therefore by (A-2),

$M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n+1}, y_{2n}, t)$ and similarly

$M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n}, y_{2n-1}, t)$

Hence $M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)$ for all n.

Now we show that

$\lim_{n\to\infty} M(y_{n+p}, y_n, t) = 1$ For all p and t > 0

Now $M(y_{n+1}, y_n, t)$

$\geq\ M(y_n, y_{n-1}, t/k)$

$\geq\ M(y_n, y_{n-2}, t/k^2)$

$\geq$ ....

$\geq\ M(y_1, y_0, t/k^n) \to 1$ As $t/k^n \to \infty$ as $n \to \infty$

Thus the result holds for $p = 1$. By induction hypothesis suppose that the result hotels for p = r, now

$M(y_n, y_{n+r+1}, t) \geq$

$M(y_n, y_{n+r}, t/2) * M(y_{n+r}, y_{n+r+1}, t/2) \to 1*1 = 1$

Thus the result holds for $p = r + 1$

Hence $\{y_n\}$ is a Cauchy sequence in X and as X is complete we get $\{y_n\} \to z \in X$.

Hence

$Ax_{2n} \to z, Sx_{2n} \to z$ ... (I)

$Tx_{2n+1} \to z, Bx_{2n+1} \to z$ .... (II)

Let S be continuous and pairs (A, S) are compatible of type(R) we get

AS $x_{2n} \to Sz, SSx_{2n} \to Sz$

From contractive condition we get

$$\phi(M(ASx_{2n}, Bx_{2n+1}, kt), M(SSx_{2n}, Tx_{2n+1}, t), M(SSx_{2n}, ASx_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, kt) M(Tx_{2n+1}, Ax_{2n}, t)) \geq 0$$

Taking limit as $n \to \infty$ we get

$$\phi(M(Sz, z, kt), M(Sz, z, t), M(Sz, Sz, t), M(z, z, kt), M(z, z, t)) \geq 0$$

$$\Rightarrow \phi(M(Sz, z, kt), M(Sz, z, t), 1, 1, 1) \geq 0$$

Since ϕ is non increasing and ψ is non decreasing in third, fourth and fifth argument therefore we can write

$$\phi(M(Sz, z, kt), M(Sz, z, t), M(Sz, z, t), M(Sz, z, t), M(Sz, z, t)) \geq 0$$

$M(Sz, z, kt) \geq M(Sz, z, t)$ and

$N(Sz, z, kt) \leq N(Sz, z, t)$

$\Rightarrow Sz = z$

Step II: Put $x = z, y = x_{2n+1}$ in condition (IV),

$$\phi(M(Az, Bx_{2n+1}, kt), M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Tx_{2n+1}, Bx_{2n+1}, kt), M(Tx_{2n+1}, Az, t) \geq 0$$

As $n \to \infty$

$$\phi(M(Az, z, kt), M(Sz, z, t), M(Sz, Az, t), M(z, z, kt), M(z, Az, t)) \geq 0$$

$\because\ Sz\ = z$

$$\phi(M(Az, z, kt), M(z, z, t), M(z, Az, t), M(z, z, kt), M(z, Az, t)) \geq 0$$

$$\phi(M(Az, z, kt), 1, M(Az, z, t), 1, M(Az, z, t)) \geq 0$$

$\Rightarrow$ Since ϕ is non-increasing and ψ is non –decreasing in second and fourth argument

$$\phi(M(Az, z, kt), M(Az, z, t), M(Az, z, t), M(Az, z, t), M(Az, z, t) \geq 0$$

$\Rightarrow\ M(Az, z, kt) \geq M(Az, z, t)$ And

$N(Az, z, kt) \leq N(Az, z, t)$

$\Rightarrow\ Az = z = Sz$

Step III: As $A(X) \subset T(X),\ \exists u \in X$

$Az\ = Tu = z$

Put $x = x_{2n}$, $y = u$ is condition (IV), we have

$$\phi(M(Ax_{2n}, Bu, kt), M(Sx_{2n}, Tu, t), M(Sx_{2n}, Ax_{2n}, t), M(Tu, Bu, kt), M(Tu, Ax_{2n}, t)\ ) \geq 0$$

Taking Limit $n \to \infty$

$$\phi(M(z, Bu, kt), M(z, Tu, t), M(z, z, t), M(Tu, Bu, kt), M(Tu, z, t)\ ) \geq 0$$

$\because\ Az\ = Tu\ = z$

$$\Rightarrow \phi(M(z, Bu, kt), M(z, z, t), M(z, z, t), M(z, Bu, kt), M(z, z, t) \geq 0$$

$$\Rightarrow \phi(M(z, Bu, kt), 1, 1, M(z, Bu, kt, 1) \geq 0$$

Since ϕ is non increasing and ψ is non decreasing in second, third and fifth argument then

$\phi(M(z,Bu,kt),M(z,Bu,t),M(z,Bu,t),$
$M(z,Bu,kt),M(z,Bu,t)\geq 0$
$\Rightarrow \phi(M(z,Bu,kt)\geq M(z,Bu,t)$ And
$\psi(N(z,Bu,kt)\leq N(z,Bu,t)$
$\Rightarrow Bu=z$
And we get $Tu=Bu=z$ and as (B, T) is compatible of type R so we get
$BTu=TBu$ i.e. $Bz=Tz$
Step IV: Take $x=z$ and $y=z$ in condition (IV), we get
$\phi(M(Az,Bz,kt),M(Sz,Tz,t),M(Sz,Az,t),$
$M(Tz,Bz,kt),M(Tz,Az,t))\geq 0$

$\phi(M(z,Bz,kt),M(z,Bz,t),M(z,z,t),$
$M(Bz,Bz,kt),M(Bz,z,t)\geq 0$

$\phi(M(z,Bz,kt),M(z,Bz,kt),1,1,M(Bz,z,t)\geq 0$
Since ϕ is non increasing in second, third and fifth argument
$\phi(M(z,Bz,kt),M(z,Bz,t),M(Bz,z,t),$
$M(Bz,z,t),M(Bz,z,t)\geq 0$
$M(z,Bz,kt)\geq M(Bz,z,t)$
Bz = z
Therefore $Bz=Tz=z$
And so we get
$Az=Bz=Sz=Tz=z$
Hence z is a common fixed point A, B, S, and T
Uniqueness: Let z and z' be two common fixed points of the maps A, B, S and T. Then
$Az=Bz=Tz=Sz=z$ And
$Az'=Bz'=Tz'=Sz'=z'$
Using condition (IV), we get
$\phi(M(Az,Bz',kt),M(Sz,Tz',t),M(Sz,Az,t),$
$M(Tz',Bz',kt),M(Tz',Az,t))\geq 0$
$\phi(M(z,z',kt),M(z,z',t),M(z,z,t),M(z',z',kt),$
$M(z',z,t))\geq 0$
$\phi(M(z,z',kt),M(z,z',t),1,1,M(z',z,t)\geq 0$
Since ϕ is none increasing in third and fourth argument so?
$\phi(M(z,z',kt),M(z,z',t),M(z,z',t),M(z,z',t),$
$M(z,z',t))\geq 0$
$\phi(M(z,z',kt),M(z,z',t),M(z,z',t),M(z,z',t),$
$M(z,z',t))\geq 0$
$M(z,z',kt)\geq M(z,z',t)$ and $z=z'$.

Hence z is a unique common fixed point maps A, B, S, T.

COROLLARY 3.1
Let A, B, S and T be self mappings of a complete fuzzy metric space $(X,M,*)$ with continuous t-norm and co t-norm defined by
$$a*b=\min\{a*b\}\{a,b\}\in[0,1]$$
Satisfying I to III
$\phi(M(Ax,By,kt),M(Sx,Ty,t),$
$M(Sx,Ax,t),M(Tx,Ay,t)\geq 0$
Then A, B, S and T have a unique common fixed point.
COROLLARY 3.2
Let A, B, S and T be self mappings of a complete fuzzy metric space $(X,M,*)$ with
Continuous t-norm defined by
$$a*b=\min\{a*b\}\,where\{a,b\}\in[0,1]$$
Satisfying i, ii, iii, v of theorem3.1 and there exist some $k\in(0,1)$ such that for all $x,y\in X,t>0$
$$\phi(M(Ax,By,kt),M(Sx,Ty,t),M(Sx,Ax,t),$$
$$M(Ty,By,2t),M(Ty,Ax,t))\geq 0$$
Then A, B, S and T have a unique common fixed point.

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REFERENCES

[1] Y.J Cho, B.K. Sharma, and D.R. Sahu, Semi-compatibility and fixed points, Math. Japonica 42(1), 91-98, 1995.

[2] Cho, on common fixed points in fuzzy metric spaces, Inter. Math. Forum 1 (10), 471-479, 2006.

[3] Zi- Ke Deng, Fuzzy pseudo metric spaces, J. Math. Anal. Appl.86, 74-95, 1982.

[4] M.A, Erceg., Metric spaces in Fuzzy set theory, J. Math. Anal. Appl., 69, 205-230, 1979.

[5] A. George and P. Veeramani, On same results of analysis for fuzzy metric spaces, Fuzzy sets and systems 90, 365-368, 1997.

[6] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27, 385-389, 1988.

[7] G. Jungck, Compatible mappings and common fixed points. Internat. J.Math. and Math. Sci., 9(4), 771-779, 1986.

[8] Kramosil and J. Machalek, Fuzzy metric and statistical metric spaces, Kybernetika 11, 336-344, 1975..

[9] M. Koireng et.al. , Common fixed point theorem of Compatibility of type R in fuzzy metric space, Gen.math.Notes, pp 58 –62, May2012.

[10] O. Kaleva and S. Seikkala, on fuzzy metric spaces, Fuzzy Sets and Systems 12, 215-229, 1984.

[11] Pathak, Chang, Cho, Fixed point theorems for compatible mappings of type (P), Indian journal of Math. 36(2), 151-166, 1994.

[12] Y Rohan ,M.R.Singh And Shambhu , Common Fixed Point Of Compatible Mapping Of Type (R) In Banach spaces,Proc.Of ath.Soci.Bhu,20, 77-87, 2004.

[13] B.Singh and M.S.Bhadauriya, Common Fixed Point Theorems for Pair of Self Mapping Satisfying Common (E. A) like Property in Fuzzy Metric Spaces, International Journal of Computer Applications 75(16) , 29-32, August 2013.

[14] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math.,10 , 314- 334. 1960.

[15] B. Singh and M.S Chouhan, Common fixed points of compatible maps in fuzzy metric spaces, fuzzy sets and systems, 115, 471-475, 2000.

[16] S. Sessa On weak commutativity condition of mappings in fixed point considerations, publ.inst. Math. Besgrad, 32 (46) 149-153. 1982.

[17] L .A Zadeh, Fuzzy sets, Inform and Control, 8, 338-353, 1965.