Unreliable Batch Arrival Retrial Queue with Negative Arrivals, Multi-Types Of Heterogeneous Service, Feedback and Randomized J Vacations

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Abstract: A batch arrival retrial queue with two types of arrival - positive and negative is considered. Positive customers arrive in batches according to Poisson process. Server provides M types of service. If the server is idle upon the arrival of a batch, one of the customers in the batch receives any one of the types immediately and others join the orbit. Otherwise, all the customers in the batch join the orbit. After completion of service, the unsatisfied customers join the orbit as feedback customers. Whenever the orbit becomes empty the server takes a vacation of random length. At each vacation completion epoch, if the system is empty, the server leaves for another vacation of same length or remains idle in the system. This pattern continues until the server finds at least one customer in the orbit or the number of vacations reaches J. At the end of Jth vacation, even if the orbit is empty the server remains in the system for new arrival. The arrival of negative customer brings the server down and makes the interrupted customer to leave the system. Using supplementary variable technique, various performance measures are derived. Stochastic decomposition property is established and particular cases are deduced.

Keywords: Retrial queue, negative customers, feedback, randomized J vacation.

1. INTRODUCTION

Queueing systems with vacation have been found to be useful in modeling the production systems, communication systems and manufacturing systems. Chang and Ke[2] introduced the concept of modified vacation policy in which on finding the empty orbit, the server takes at most J vacations repeatedly until at least one customer is in the orbit. Jain and Charu Bhargava [8] suggested an M/G/1 retrial queueing system with modified vacation and K phase repair and obtained the probability generating function of the steady state queue size at random epoch. Chen et al.[3] investigated a retrial queue with modified vacation and server breakdown. Rajadurai et al.[10] analysed the steady state batch arrival retrial queueing system with two types of service under modified vacation policy.

In recent years there has been a rapid increase in the literature on retrial queueing system with negative customers. Queue with negative arrivals called G-queue was introduced by Gelenbe [4] with a view to modeling neural networks. The arrival of negative customer brings the server down and makes the interrupted customer to leave the system. For a comprehensive analysis of queueing systems with negative arrivals, reader may refer to Gelenbe[5-7] and Artalejo[1]. Liu et al. [9] analysed an M/G/1 retrial G-queue with preemptive resume and feedback under N-policy vacation. Wu and Yin [11] considered an unreliable M/G/1 retrial G-queue with non exhaustive random vacation and derived steady state solution for both queueing measures and reliability quantities.

In this paper, batch arrival retrial queue with negative customers, multi-types of service and feedback is analyzed under randomized J vacation policy. Under this policy, at the end of each vacation if no customers are waiting for service, the server leaves for another vacation with certain probability or remains in the system with complementary probability.

2. MODEL DESCRIPTION

Single server queueing system with two types of arrivals - positive and negative is considered. Positive customers arrive in batches according to Poisson process with rate λ. At every arrival epoch, a batch of k customers arrive with probability Ck. The generating function of the sequence {Ck} is C(z) with first two moments m1 and m2. The server provides M types of service. Customers opt the i\textsuperscript{th} type of service with probability pi (1 ≤ i ≤ M). There is no waiting space in front of the server and therefore if the arriving batch of positive customers finds the server idle, then one of the customers receives service and the others join the orbit. After receiving service, the customer may again join the orbit as a feedback customer with probability δ or depart the system with its complementary probability δ (=1-δ).

At a service completion epoch if the orbit is empty, the server leaves for a vacation of random length V. At a vacation completion epoch, if the orbit is still empty the server remains idle in the system with
The stochastic behavior of the retrial queueing system can be described by the Markov process \( \{ X(t), t \geq 0 \} \) where \( C(t) \) denotes the server state 0, I, M+i, 2M+j according as the server being idle, busy in \( i^{th} \) type service, under repair which is failed during \( i^{th} \) type service or on \( i^{th} \) vacation. \( N(t) \) corresponds to the number of customers in the orbit. If \( C(t) = 0 \), then \( \xi(t) \) represents the elapsed retrial time. If \( C(t) = i \), then \( \xi(t) \) represents the elapsed type \( i \) service time. If \( C(t) = M+i \), then \( \xi(t) \) represents the elapsed repair time of the server failed during \( i^{th} \) type service, \( (1 \leq i \leq M) \). If \( C(t) = 2M+j \), then \( \xi(t) \) represents the elapsed \( j^{th} \) vacation time \( (1 \leq j \leq J) \).

### 3. STEADY STATE ANALYSIS

For the process \( \{ X(t), t \geq 0 \} \), define the probability densities

\[
I_0(t) = P \{ C(t) = 0, N(t) = n \} \\
I_i(x,t) = P \{ C(t) = i, N(t) = n, x \leq \xi(t) < x+dx \}, x \geq 0, n \geq 1 \ \\
P_{R_{ij}}(x,t)dx = P \{ C(t) = i+M+j, N(t) = n, x \leq \xi(t) < x+dx \}, x \geq 0, n \geq 0, \\
I_{R_{ij}}(x,t)dx = P \{ C(t) = i+M+j, N(t) = n, x \leq \xi(t) < x+dx \}, x \geq 0, n \geq 0.
\]

The system of equilibrium equations that governs the model under consideration are

\[
\lambda^+I_0 = \int_0^\infty V_0^{(j)}(x)\gamma(x)dx + q \sum_{j=1}^\infty \int_0^\infty V_0^{(j)}(x)\gamma(x)dx
\]

\[
\frac{d}{dx}I_n^{(i)}(x) = -\lambda x^{+} + \eta(x)I_n^{(i)}(x), n \geq 1
\]

\[
\frac{d}{dx}P_n^{(i)}(x) = -\lambda^{+} + \mu(x)P_n^{(i)}(x) + (1-\delta_{00})\lambda^{+} \sum_{k=1}^{n} C_kP_n^{(i-k)}(x), \\
n \geq 0, i = 1,2,...,M
\]

\[
\frac{d}{dx}R_{n}^{(i)}(x) = -\lambda^{+} + \beta_1(x)R_{n}^{(i)}(x) + (1-\delta_{00})\lambda^{+} \sum_{k=1}^{n} C_kV_{n}^{(i-k)}(x), \\
n \geq 0, i = 1,2,...,M
\]

with boundary conditions

\[
I_0(0) = \sum_{i=1}^{M} \delta^i P_{R_{i0}}^{-1}(0)I_{0}(x)dx + \int_0^\infty \gamma(x)I_0(x)dx + \int_0^\infty \gamma(x)I_0^1(x)dx
\]

\[
P_{n}^{(i)}(0) = P_i \left[ \lambda^{+} + I_0 + \sum_{k=1}^{n} C_k \right] \int_0^\infty \gamma(x)I_0^k(x)dx, n \geq 1, i = 1,2,...,M
\]

\[
R_{n}^{(i)}(0) = \lambda^{+} \int_0^\infty V_{n}^{(j)}(x)\gamma(x)dx, n \geq 0, i = 1,2,...,M
\]

\[
V_{n}^{(j)}(0) = \left\{ \begin{array}{ll}
\int_0^\infty V_{n}^{(j-1)}(x)\gamma(x)dx, & n = 0, j = 2,3,...,J \\
\int_0^\infty V_{n}^{(j-1)}(x)\gamma(x)dx, & n \neq 0, j = 2,3,...,J
\end{array} \right.
\]

\[
V_{n}^{(j)}(0) = \left\{ \begin{array}{ll}
\int_0^\infty V_{n}^{(j-1)}(x)\gamma(x)dx, & n = 0, j = 2,3,...,J \\
\int_0^\infty V_{n}^{(j-1)}(x)\gamma(x)dx, & n \neq 0, j = 2,3,...,J
\end{array} \right.
\]

Define the following probability generating functions for \(|x| \leq 1\)

\[
I(x) = \sum_{n=1}^{\infty} I_n(x)z^n, \quad P_{n}^{(i)}(x,z) = \sum_{n=0}^{\infty} P_{n}^{(i)}(x)z^n,
\]

\[
R_{n}^{(i)}(x,z) = \sum_{i=1}^{M} R_{n}^{(i)}(x)z^n, i = 1,2,...,M \quad \text{and}
\quad V_{n}^{(j)}(x,z) = \sum_{n=0}^{\infty} V_{n}^{(j)}(x)z^n, j = 1,2,...,J
\]

### 3.1. Theorem

The joint steady state distributions of the server state and orbit size are given by

\[
I(x) = I_0(1-A^+(\lambda^+)) \left[ C(x) + \sum_{i=1}^{M} \delta^i B_{i}^{j} R_{n}^{(i)}(g(x)) \right] \int_0^x I_0^1(x)dx
\]

\[
A^+(\lambda^+) = \frac{\lambda^{+}(1-B_{i}^{j}(g(x))R_{n}^{(i)}(h(x))) + \gamma(x)N(V^{(1)}(x)-1)}{\lambda^{+}(1-B_{i}^{j}(g(x))R_{n}^{(i)}(h(x))) + \gamma(x)N(V^{(1)}(x)-1)}
\]
Multiplying equations (2) to (11) by $z^n$, we get the following partial differential equations:

\[
P^{(i)}(z) = I_0 + p_1 [1 - B^*_1(g(z))] A^*(\lambda^+)(C(z) - 1) +
\]

\[
[A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))] \int [N(V*(h(z)) - 1)] / D(z),
\]

\[1 \leq i \leq M \]

(13)

\[
R^{(i)}(z) = I_0 - p_1 [1 - B^*_1(g(z))] [1 - B^*_1(h(z))] A^*(\lambda^+)(C(z) - 1) +
\]

\[
[A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))] \int [N(V*(h(z)) - 1)] / [D(z)D(z)],
\]

\[1 \leq i \leq M \]

(14)

Further integration with respect to $x$ from 0 to $\infty$, we have

\[
\int_0^\infty [V^*(\lambda^+)](x)dx = \int_0^\infty [V^*(\lambda^+)](0)dx
\]

Solving equation (5) at $n=0$, we have

\[
V^0_j(0) = V^0_j(0) e^{-\lambda^+ x(1 - V(x))}
\]

Multiplying equation (24) by $\gamma(x)$ and integrating with respect to $x$ from 0 to $\infty$, we have

\[
\int_0^\infty [V^0_j(0)\gamma(x)](x)dx = \int_0^\infty [V^0_j(0)\gamma(x)](0)dx
\]

Equation (10) gives

\[
V^0_j(0) = \bar{V} V^0_j(0) V^*(\lambda^+), j = 2,3,..,J
\]

From equations (10) and (11) it is clear that

\[
V^0_J(0) = V^0_J(0)
\]

Applying equations (26) repeatedly for $j = J - 1, .., j$, we get

\[
V^0_j(0) = \frac{\lambda^+ I_0}{[\bar{V} V^*(\lambda^+) J - 1] + \frac{qI(1 - (\bar{V} V^*(\lambda^+) J - 1))}{(\bar{V} V^*(\lambda^+) J - 1)(1 - (\bar{V} V^*(\lambda^+) J - 1))}}
\]

Solving the partial differential equations (17), (18), (19) and (20), we get respectively

\[
V^0_j(0) = 0, j = 1,2,..,M
\]

\[
R^{(i)}(0, z) = \int_0^\infty [P^{(i)}(x,z)dx], i = 1,2,..,M
\]

(23)

(24)

(25)

(26)

(27)

(28)

(29)

(30)

(31)

(32)

(33)

(34)
Solving equation (34) by using equation (35), we obtain
\[ P^{(i)}(0, z) = I_0 \mu_i^+ p_i g(z) A^+(\lambda^+) C(z) - 1 + \frac{[A^+(\lambda^+) + C(z)](1 - A^+(\lambda^+)) |N| V^*(h(z)) - 1}{D(z)} \] (36)

Using equations (36) in equation (33) and simplifying we get
\[ R^{(i)}(0, z) = I_0 \mu_i^+ \gamma_i [1 - B_1^i(g(z))] A^+(\lambda^+) C(z) - 1 + \frac{[A^+(\lambda^+) + C(z)](1 - A^+(\lambda^+)) |N| V^*(h(z)) - 1}{D(z)} \] (37)

Using the expressions of I(0, z), P^{(i)}(0, z), R^{(i)}(0, z) and V^{(i)}(0, z), the probability generating functions I(x, z), P^{(i)}(x, z), R^{(i)}(x, z) and V^{(i)}(x, z) can be expressed in terms of I_0. Integrating the resultant expressions with respect to x from 0 to \( \infty \), we get I(z), P^{(i)}(z), R^{(i)}(z) and V^{(i)}(z) as in equations (12), (13), (14) and (15). The expression of I_0 given in equation (16) can be obtained by using the normalizing condition.

4. PERFORMANCE MEASURES

Probability that the server is idle in the non-empty system is given by
\[ I(z) = \lim_{z \to 1} I(z) = \frac{\lambda^+ M \sum_{i=1}^{M} p_i I_{1}(1 - B_1^i(\lambda^-))}{\lambda^- [1 - \delta \sum_{i=1}^{M} p_i B_1^i(\lambda^-)]} \] (38)

Probability that the server is busy is given by
\[ P(z) = \lim_{z \to 1} P^{(i)}(z) = \frac{\lambda^+ M \sum_{i=1}^{M} \lambda_i^+ m_i p_i [1 - B_1^i(\lambda^-)]}{\lambda^- [1 - \delta \sum_{i=1}^{M} p_i B_1^i(\lambda^-)]} \] (39)

Probability that the server is under repair is given by
\[ R(z) = \lim_{z \to 1} R^{(i)}(z) = \frac{\lambda^+ M \sum_{i=1}^{M} p_i B_1^i(1 - B_1^i(\lambda^-))}{[1 - \delta \sum_{i=1}^{M} p_i B_1^i(\lambda^-)]} \] (40)

Probability that the server is on vacation is given by
\[ V(z) = \lim_{z \to 1} \sum_{j=1}^{J} V^{(j)}(z) \]

\[ V^+(N) = [\lambda^+ \mu^+ \gamma_i [1 - B_1^i(g(z))] A^+(\lambda^+) C(z) - 1 + \frac{[A^+(\lambda^+) + C(z)](1 - A^+(\lambda^+)) |N| V^*(h(z)) - 1}{D(z)}] \]

\[ \lambda^+ \mu^+ \gamma_i [1 - B_1^i(g(z))] A^+(\lambda^+) C(z) - 1 + \frac{[A^+(\lambda^+) + C(z)](1 - A^+(\lambda^+)) |N| V^*(h(z)) - 1}{D(z)} \] (41)

Probability generating function of the number of customers in the orbit is given by
\[ P_q(z) = I_0(1 + I(z) + \sum_{i=1}^{M} P^{(i)}(z) + R^{(i)}(z)) + \sum_{j=1}^{J} V^{(j)}(z) \]

Mean number of customers in the orbit \( L_q \) is given by
\[ L_q = \lim_{z \to 1} \frac{1}{dz} P_q(z) \]

Let \( N(r) \) and \( D(r) \) be the numerator and denominator of \( P_q(z) \). Since \( N(q) = D(q) = N(r) = D(r) = 0 \), applying L'Hospital rule we get
\[ L_q = \frac{\lambda^+ \mu^+ \gamma_i [1 - B_1^i(g(z))] A^+(\lambda^+) C(z) - 1 + \frac{[A^+(\lambda^+) + C(z)](1 - A^+(\lambda^+)) |N| V^*(h(z)) - 1}{D(z)} \] (42)

\[ L_q = \sum_{i=1}^{M} p_i B_1^i(1 - B_1^i(\lambda^-)) \] (43)

where
\[ N(r) = N_0(2 \lambda^- - \lambda^+ m_1 A^+(\lambda^+) + \lambda^+ \gamma_i N)[1 - \delta \sum_{i=1}^{M} p_i B_1^i(\lambda^-)] \]

\[ N(r) = I_0(1 + I(z) + \sum_{i=1}^{M} P^{(i)}(z) + \sum_{j=1}^{J} V^{(j)}(z)) \]

\[ N(r) = \lambda^+ \mu^+ \gamma_i [1 - B_1^i(g(z))] A^+(\lambda^+) C(z) - 1 + \frac{[A^+(\lambda^+) + C(z)](1 - A^+(\lambda^+)) |N| V^*(h(z)) - 1}{D(z)} \]
Mean number of customers in the orbit $L_s$ is given by

$$L_s = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$

Let $N_r(z)$ be the numerator of $P_s(z)$.

$$L_s = P_s'(1) = \left[3\lambda^+ M + \lambda^+ \lambda_1 N[1 - \delta \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)]\right] I_0$$

where

$$N_r(1) = -2\lambda^+ M \lambda^+ \lambda_1 N[1 - \delta \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)] I_0$$

The availability of the server is given by

$$A(t) = \frac{1}{1 + \lambda^+ \lambda_1 N[1 - \delta \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)] I_0}$$

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The probability generating function $\chi(z)$ of the $N$-state retrial process with $N$ types of heterogeneous customers, multi-types of heterogeneous service and feedback and the other is the mean number of customers in the orbit (Ls) given that the server is idle or on vacation.

$$\chi(z) = \frac{I_0 + I(z) + \frac{1}{V} \psi(z)}{I_0}$$

The availability of the server is given by

$$A(t) = \frac{1}{1 + \lambda^+ \lambda_1 N[1 - \delta \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)] I_0}$$

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The probability generating function $\pi(z)$ of the number of customers in the classical queueing system with negative customers, server breakdown and feedback is

$$\pi(z) = \frac{I_0(z) - \lambda^+ M - \lambda^+ \lambda_1 N[1 - \delta \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)] I_0}{(1 - C(z))}$$

$$\lambda^+ \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)$$

6. STOCHASTIC DECOMPOSITION

6.1. Theorem: The number of customers in the system (Ls) can be expressed as the sum of two independent random variables, one of which is the mean number of customers (Ls) in the unreliable batch arrival G-queue with multi-type of heterogeneous service and feedback and the other is the mean number of customers in the orbit (Ls) given that the server is idle or on vacation.

**Proof:** The probability generating function $\pi(z)$ of the number of customers in the classical queueing system with negative customers, server breakdown and feedback is

$$\pi(z) = \frac{I_0(z) - \lambda^+ M - \lambda^+ \lambda_1 N[1 - \delta \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)] I_0}{(1 - C(z))}$$

$$\lambda^+ \sum_{i=1}^{M} P_i B_i \lambda^+(\lambda^-)$$

7. CONCLUSIONS

In this paper, a single server batch arrival retrial queue with negative customers, multi-types of heterogeneous service with feedback and randomized J vacation is analyzed. Using supplementary variable technique, analytical expressions for various performance measures are derived. Stochastic decomposition property is also verified. Numerical results are carried out to study the effect of some key parameters on the performance measures of the model.

**REFERENCES**


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