

ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION

$$4(x^2 + y^2) - 3xy = 16z^2$$

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Abstract -

The ternary homogeneous quadratic equation given by $4(x^2 + y^2) - 3xy = 16z^2$ representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramided numbers are presented.

Key Words: Ternary quadratic, integer solutions, figurate numbers, homogeneous quadratic, polygonal number, pyramidal numbers.

1. INTRODUCTION:

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-22] for quadratic equations with three unknowns. This communication concerns with yet another interesting

equation $4(x^2 + y^2) - 3xy = 16z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2.METHOD OF ANALYSIS:

The ternary quadratic diophantine equation to be solved is

$$4(x^2 + y^2) - 3xy = 16z^2 \quad (1)$$

To start with, it is seen that (1) is satisfied by the following integer triples

$$(x, y, z): (48, -16, 28), (0, -16, 8)$$

However, we have other choices of solutions to (1) which are illustrated below:

The substitution of the linear transformations

$$x = u+v ; y = u-v \quad (u \neq 0, v \neq 0) \quad (2)$$

in (1) leads to

$$5u^2 + 11v^2 = 16z^2 \quad (3)$$

Take

$$z = z(a, b) = 5a^2 + 11b^2 \quad (4)$$

where a, b are non-zero distinct integers.

Different patterns of solutions of (1) are illustrated below

PATTERN: 1

Write (16) as

$$16 = (\sqrt{5} + i\sqrt{11}) * (\sqrt{5} - i\sqrt{11}) \tag{5}$$

Substituting (4),(5) in (3) and employing the method of factorization, we've

$$(\sqrt{5}u + i\sqrt{11}v)(\sqrt{5}u - i\sqrt{11}v) = (\sqrt{5} + i\sqrt{11})(\sqrt{5} - i\sqrt{11}) * (\sqrt{5}a + i\sqrt{11}b)^2 (\sqrt{5}a - i\sqrt{11}b)^2$$

Equating the positive and negative factors, we get

$$(\sqrt{5}u + i\sqrt{11}v) = (\sqrt{5} + i\sqrt{11})(\sqrt{5}a + i\sqrt{11}b)^2 \tag{6}$$

$$(\sqrt{5}u - i\sqrt{11}v) = (\sqrt{5} - i\sqrt{11})(\sqrt{5}a - i\sqrt{11}b)^2 \tag{7}$$

Equating the real and imaginary parts in (6)

$$u = u(a, b) = 5a^2 - 11b^2 - 22ab$$

$$v = v(a, b) = 5a^2 - 11b^2 + 10ab$$

Substituting the values of u and v in (2), we gets

$$x = x(a, b) = 10a^2 - 22b^2 - 12ab \tag{8}$$

$$y = y(a, b) = -32ab \tag{9}$$

Thus (8),(9) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

$$(1) \quad x(2a,1) - 40pr_a \equiv -22(\text{mod } 64)$$

$$(2) \quad x(a,2) - 10pr_a \equiv -88(\text{mod } 34)$$

$$(3) \quad x(a,1) - y(a,1) + z(a,1) - 15pr_a \equiv -11(\text{mod } 15)$$

$$(4) \quad [6z(a,-a) + 3z(a,-a)] \text{ is a perfect square.}$$

$$(5) \quad x(2a,2) - 40pr_a \equiv -88(\text{mod } 88)$$

PATTERN: 2

Write (16) as

$$16 = (-\sqrt{5} + i\sqrt{11}) * (-\sqrt{5} - i\sqrt{11}) \tag{10}$$

Substituting (4),(10) in (3) and employing the method of factorization, we've

$$(\sqrt{5}u + i\sqrt{11}v)(\sqrt{5}u - i\sqrt{11}v) = (-\sqrt{5} + i\sqrt{11})(-\sqrt{5} - i\sqrt{11}) * (\sqrt{5}a + i\sqrt{11}b)^2 (\sqrt{5}a - i\sqrt{11}b)^2$$

Equating the positive and negative factors, we get

$$(\sqrt{5}u + i\sqrt{11}v) = (-\sqrt{5} + i\sqrt{11})(\sqrt{5}a + i\sqrt{11}b)^2 \tag{11}$$

$$(\sqrt{5}u - i\sqrt{11}v) = (-\sqrt{5} - i\sqrt{11})(\sqrt{5}a - i\sqrt{11}b)^2 \tag{12}$$

Equating the real and imaginary parts in (11)

$$u = u(a, b) = -5a^2 + 11b^2 - 22ab$$

$$v = v(a, b) = 5a^2 - 11b^2 - 10ab$$

Substituting the values of u and v in (2), we get

$$x = x(a, b) = -32ab \tag{13}$$

$$y = y(a, b) = -10a^2 + 22b^2 - 12ab \tag{14}$$

Thus (13),(14) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

- (1) $y(-2a,3) + 40pr_a \equiv 198(\text{mod } 112)$
- (2) $y(2,b) - 22pr_b \equiv -40(\text{mod } 46)$
- (3) $x(2a,2) - y(2a,2) - z(2a,a) - 20pr_a \equiv -132(\text{mod } 100)$
- (4) $[5z(b,b) - 4z(b,b)]$ is a perfect square.
- (5) $y(3,a) - 22pr_a \equiv -90(\text{mod } 58)$

PATTERN:3

Substituting the linear transformations

$$\begin{aligned} u &= x - 11T \\ v &= x + 5T \end{aligned} \quad (*)$$

in (3), we get

$$z^2 = x^2 + 55T^2 \quad (15)$$

which is satisfied by

$$T = 2pq \quad (16)$$

$$z = 55p^2 + q^2 \quad (17)$$

$$x = p^2 - 55q^2 \quad (18)$$

From (*), we get,

$$u = p^2 - 55q^2 - 22pq \quad (19)$$

$$v = p^2 - 55q^2 + 10pq \quad (20)$$

Substituting the values of u and v in (2), we've

$$\begin{aligned} x &= x(p, q) \\ &= 2p^2 - 110q^2 - 12pq \end{aligned} \quad (21)$$

$$y = y(p, q) = -32pq \quad (22)$$

Thus (21) (22) and (17) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

- (1) $x(2,a) + 110pr_a \equiv 8(\text{mod } 86)$
- (2) $x(a,1) - 2pr_a \equiv -110(\text{mod } 14)$
- (3) $x(a,1) - y(a,1) + z(a,1) - 57pr_a \equiv -109(\text{mod } 37)$
- (4) $[10z(q,q) + 4z(q,q)]$ is a perfect square.
- (5) $x(a,a) - y(a,a) + 152t_{4,a} = 0$

PATTERN:4

Write equation (15) as

$$(z+x)(z-x) = (11T)(5T) \quad (23)$$

The above equation is written in the form of ratio as

$$\frac{z+x}{11T} = \frac{5T}{z-x} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (24)$$

The equation (24) is equivalent to the following two equations

$$\beta x - 11\alpha T + \beta z = 0 \quad (25)$$

$$-\alpha x - 5\beta T + \alpha z = 0 \quad (26)$$

Applying the method of cross multiplication, we get,

$$\frac{x}{-11\alpha^2 + 5\beta^2} = \frac{T}{-\alpha\beta - \alpha\beta} = \frac{z}{-5\beta^2 - 11\alpha^2}$$

Therefore ,

$$X = x(\alpha, \beta) = -11\alpha^2 + 5\beta^2$$

$$T = T(\alpha, \beta) = -2\alpha\beta$$

$$z = z(\alpha, \beta) = -(11\alpha^2 + 5\beta^2) \quad (27)$$

Substituting in the values of X and T in (*), we've

$$u = u(\alpha, \beta) = 11\alpha^2 + 5\beta^2$$

$$v = v(\alpha, \beta) = 11\alpha^2 + 5\beta^2$$

Substituting in the values of u and v in (2), we've

$$x = x(\alpha, \beta) = 22\alpha^2 + 10\beta^2 \quad (28)$$

$$y = y(\alpha, \beta) = 0 \quad (29)$$

Thus (28),(29) and (27) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

$$(1) \quad 2z(-1, \alpha) - z(-1, \alpha) + 11 + 5t_{4,\alpha} = 0$$

$$(2) \quad x(\beta, \beta) - z(\beta, \beta) - 16t_{4,\beta} = 0$$

$$(3) \quad z(2\alpha, 2\alpha) + 64t_{4,\alpha} = 0$$

$$(4) \quad x(\beta, \beta) - 2z(\beta, \beta) - 32t_{4,\beta} = 0$$

$$(5) \quad x(2, \alpha) - z(2, \alpha) - 15pr_\alpha \equiv -44(\text{mod } 9)$$

PATTERN: 5

Write equation (15) as

$$(z + x)(z - x) = (11T)(5T) \quad (30)$$

The above equation is written in the form of ratio as

$$\frac{z - x}{11T} = \frac{5T}{z + x} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (31)$$

The equation (31) is equivalent to the following two equation

$$- \beta x - 11\alpha T - \beta z = 0 \quad (32)$$

$$- \alpha x + 5\beta T - \alpha z = 0 \quad (33)$$

Applying the method of cross multiplication, we get,

$$\frac{x}{11\alpha^2 - 5\beta^2} = \frac{T}{-\alpha\beta - \alpha\beta} = \frac{z}{-5\beta^2 - 11\alpha^2}$$

Therefore ,

$$x = x(\alpha, \beta) = 11\alpha^2 - 5\beta^2$$

$$T = T(\alpha, \beta) = -2\alpha\beta$$

$$z = z(\alpha, \beta) = -(11\alpha^2 + 5\beta^2) \quad (34)$$

Substituting the values of x and T in (**), we've

$$u = u(\alpha, \beta) = -11\alpha^2 + 5\beta^2 + 22\alpha\beta$$

$$v = v(\alpha, \beta) = -11\alpha^2 + 5\beta^2 - 10\alpha\beta$$

Substituting in the values of u and v in (2), we've

$$x = x(\alpha, \beta) = -22\alpha^2 + 10\beta^2 + 12\alpha\beta \quad (35)$$

$$y = y(\alpha, \beta) = 32\alpha\beta \quad (36)$$

Thus (35),(36) and (34) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

$$(1) \quad x(\alpha, \alpha) - z(\alpha, \alpha) - 40t_{4,\alpha} = 0$$

$$(2) \quad x(3, \beta) + 10pr_\beta \equiv 198(\text{mod } 46)$$

$$(3) \quad [-z(6\beta, 6\beta)] \text{ is a perfect square.}$$

$$(4) \quad x(\alpha, 1) + y(\alpha, 1) + z(\alpha, 1) - 11pr_\alpha \equiv -15(\text{mod } 33)$$

$$(5) \quad 3z(\beta, 1) - 2z(\beta, 1) + 11t_{4,\beta} + 5 = 0$$

PATTERN: 6

Write equation (15) as

$$(z + x)(z - x) = (55T)(T) \quad (37)$$

The above equation is written in the form of ratio as

$$\frac{z + x}{55T} = \frac{T}{z - x} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (38)$$

The equation (38) is equivalent to the following two equations

$$\beta x - 55\alpha T + \beta z = 0 \quad (39)$$

$$\alpha x + \beta T - \alpha z = 0 \quad (40)$$

Applying the method of cross multiplication, we get,

$$\frac{x}{55\alpha^2 - \beta^2} = \frac{T}{2\alpha\beta} = \frac{z}{55\alpha^2 + \beta^2}$$

Therefore ,

$$x = x(\alpha, \beta) = 55\alpha^2 - \beta^2$$

$$T = T(\alpha, \beta) = 2\alpha\beta$$

$$z = z(\alpha, \beta) = 55\alpha^2 + \beta^2 \quad (41)$$

Substituting the values of x and T in (**), we've

$$u = u(\alpha, \beta) = 55\alpha^2 - \beta^2 - 22\alpha\beta$$

$$v = v(\alpha, \beta) = 55\alpha^2 - \beta^2 + 10\alpha\beta$$

Substituting in the values of u and v in (2), we've

$$\begin{aligned} x &= x(\alpha, \beta) \\ &= 110\alpha^2 - 2\beta^2 - 12\alpha\beta \end{aligned} \quad (42)$$

$$y = y(\alpha, \beta) = -32\alpha\beta \quad (43)$$

Thus (42),(43) and (41) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

$$(1) \quad x(\alpha, \alpha) - y(\alpha, \alpha) - 128t_{4,\alpha} = 0$$

$$(2) \quad x(\beta, -6) - 110pr_\beta \equiv -72(\text{mod } 322)$$

$$(3) \quad [z(2\beta, 6\beta)] \text{ is a perfect square.}$$

$$(4) \quad \begin{aligned} x(2\beta, 1) - y(2\beta, 1) - z(2\beta, 1) - 220pr_\beta \\ \equiv -3(\text{mod } 180) \end{aligned}$$

$$(5) \quad 2z(1, \beta) - z(1, \beta) - t_{4,\beta} - 55 = 0$$

REMARK:

In addition to (24),(31),(38) and (15) may also be expressed in the form of ratios in four different ways that are presented below:

WAY1:

$$\frac{z + x}{5T} = \frac{11T}{z - x} = \frac{\alpha}{\beta}$$

WAY2:

$$\frac{z - x}{5T} = \frac{11T}{z + x} = \frac{\alpha}{\beta}$$

WAY3:

$$\frac{z - x}{T} = \frac{55T}{z + x} = \frac{\alpha}{\beta}$$

Way4:

$$\frac{z - x}{T^2} = \frac{55}{z + x} = \frac{\alpha}{\beta}$$

Solving each of the above system of equations by following the procedure presented in pattern (4),(5),(6), the corresponding integer solutions to (1) are found to be as given below:

Solution for way 1:

$$x = x(\alpha, \beta) = 10\alpha^2 - 22\beta^2 - 12\alpha\beta$$

$$y = y(\alpha, \beta) = -32\alpha\beta$$

$$z = z(\alpha, \beta) = 5\alpha^2 + 11\beta^2$$

Solution for way 2:

$$x = x(\alpha, \beta) = 10\alpha^2 - 22\beta^2 + 12\alpha\beta$$

$$y = y(\alpha, \beta) = 32\alpha\beta$$

$$z = z(\alpha, \beta) = -(5\alpha^2 + 11\beta^2)$$

Solution for way 3:

$$x = x(\alpha, \beta) = 2\alpha^2 - 110\beta^2 + 12\alpha\beta$$

$$y = y(\alpha, \beta) = 32\alpha\beta$$

$$z = z(\alpha, \beta) = -\alpha^2 - 55\beta^2$$

Solution for way 4:

$$x = x(\alpha, \beta) = 2\alpha^2 - 110\beta^2 - 12\alpha\beta$$

$$y = y(\alpha, \beta) = -32\alpha\beta$$

$$z = z(\alpha, \beta) = \alpha^2 + 55\beta^2$$

3.CONCLUSION:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by

$$4(x^2 + y^2) - 3xy = 16z^2$$

As quadratic equations are rich in variety, one may search for other choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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