ON THE BINARY QUADRATIC DIOPHANTINE EQUATION

\[ x^2 - 3xy + y^2 + 18x = 0 \]

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Abstract: The binary quadratic equation \[ x^2 - 3xy + y^2 + 18x = 0 \] represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Key Words: Binary quadratic equation, Integral solutions.

MSC subject classification: 11D09.

1. INTRODUCTION:

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [¹–⁶]. In [⁷–¹⁶] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by \[ x^2 - 3xy + y^2 + 18x = 0. \] The recurrence relations satisfied by the solutions \( x \) and \( y \) are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS:

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

\[ x^2 - 3xy + y^2 + 18x = 0 \]  \quad (1)

Note that (1) is satisfied by the following non-zero integer pairs

(18,18), (18,36), (-18,-54), (36,72), (-54,-162).

However, we have solutions for (1), which are illustrated below:

Solving (1) for \( x \), we've

\[ x = \frac{1}{2} \left[ (3y - 18) \pm \sqrt{5y^2 - 108y + 324} \right] \]  \quad (2)

Let \( \alpha^2 = 5y^2 - 108y + 324 \)

which is written as

\[ (5y - 54)^2 = 5\alpha^2 + 36^2 \]

\[ \Rightarrow Y^2 = 5\alpha^2 + 36^2 \]  \quad (3)

Where

\[ Y = 5y - 54 \]  \quad (4)

The least positive integer solution of (3) is
\[ \alpha_0 = 144, \ y_0 = 324 \]

Now, to find the other solution of (3), consider the Pellian equation
\[ Y^2 = 5\alpha^2 + 1 \]  
whose fundamental solution is \((\tilde{\alpha}_0, \tilde{Y}_0) = (4,9)\)

The other solutions of (5) can be derived from the relations
\[ \tilde{Y}_n = \frac{f_n}{2} \quad \tilde{\alpha}_n = \frac{g_n}{2\sqrt{15}} \]

where
\[ f_n = [(9 + 4\sqrt{15})^{n+1} + (9 - 4\sqrt{15})^{n+1}] \]
\[ g_n = [(9 + 4\sqrt{15})^{n+1} + (9 - 4\sqrt{15})^{n+1}] \]

Applying the lemma of Brahmagupta between \((\alpha_0, Y_0)\) & \((\tilde{\alpha}_n, \tilde{Y}_n)\),
the other solutions of (3) can be obtained from the relations
\[ \alpha_{n+1} = 72f_n + \frac{162g_n}{\sqrt{5}} \]  
\[ Y_{n+1} = 162f_n + 72\sqrt{5}g_n \]  

Taking positive sign on the R.H.S of (2) and using (4), (6) & (7), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows,
\[ x_{n+1} = \frac{1}{2}(3Y_{n+1} - 18 \pm \alpha_{n+1}) \]  
\[ y_{n+1} = \frac{1}{5}(3Y_{n+1} + 54) \]  

The recurrence relations satisfied by \(x_{n+1}, y_{n+1}\) are respectively
\[ x_{n+5} - 322x_{n+3} + x_{n+1} = -4608 \]
\[ y_{n+5} - 322y_{n+3} + y_{n+1} = -3456 \]

A few numerical examples are presented in the table below.

<table>
<thead>
<tr>
<th>n</th>
<th>(x_{n+1})</th>
<th>(y_{n+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3042</td>
<td>1170</td>
</tr>
<tr>
<td>2</td>
<td>977202</td>
<td>373266</td>
</tr>
<tr>
<td>4</td>
<td>314651394</td>
<td>120187026</td>
</tr>
</tbody>
</table>

A few interesting relations among the solutions are presented below.

1) \(x_{n+1}\) & \(y_{n+1}\) are always even
2) \(x_{n+1} = 0 (\text{mod} \ 2)\)  
3) \(y_{n+1} = 0 (\text{mod} \ 2)\)
4) \(\frac{3}{81}[945y_{2n+2} - 360x_{2n+2} - 7614] + 12\) is a nasty number.
5) \(\frac{1}{162}[945y_{2n+2} - 360x_{2n+2} - 7614] + 2\) is a quadratic integer.
is a cubic integer.

\[
\frac{1}{162} \left[ 945y_{n+3} - 360x_{n+3} - 7614 \right] + \\
3 \left[ \frac{1}{162} (945y_{n+1} - 360x_{n+1} - 7614) \right]
\]

Remarkable observations:

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the hyperbola.

\[
131220U^2 - V^2 = 524880
\]

where

\[
U = \frac{1}{162} (945y_{n+1} - 360x_{n+1} - 7614)
\]

\[
V = 810x_{n+1} - 2115y_{n+1} + 17010
\]

103680 \(U_1^2 - V_1^2 = 414720\)

where

\[
U_1 = \frac{1}{162} (945y_{n+1} - 360x_{n+1} - 7614)
\]

\[
V_1 = \frac{1}{144} (5y_{n+3} - 1605y_{n+1} + 17280)
\]

2) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the parabola.

\[
N^2 = 162M - 52488
\]

Where

\[
M = 945y_{2n+2} - 360x_{2n+2} - 7614
\]

\[
N = 810x_{n+1} - 2115y_{n+1} + 17010
\]

\[
N_i^2 = 640M_i - 207360
\]
M = 945y_{2n+2} - 360x_{2n+2} - 7614

N_1 = -1605y_{n+1} + 5y_{n+3} + 17280

**CONCLUSION:**

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

**Acknowledgement:**

The financial support from the UGC, New Delhi (F.MRP-5122/14 (SERO/UGC) dated March 2014) for a part of this work is gratefully acknowledged.

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