COMPUTATION OF SOME WONDERFUL RESULTS INVOLVING CERTAIN POLYNOMIALS

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Abstract - *In this paper we have established some indefinite integrals involving certain polynomials in the form of Hypergeometric function .The results represent here are assume to be new.*

Key Words: Hypergeometric function, Lucus Polynomials, Gegenbaur Polynomials, Hermonic number, Bernoulli Polynomials, and Hermite Polynomials.

1. Introduction

The special function is one of the central branches of Mathematical sciences initiated by L'Euler .But systematic study of the Hypergeometric functions were initiated by C.F Gauss, an imminent German Mathematician in 1812 by defining the Hypergeometric series and he had also proposed notation for Hypergeometric functions. Since about 250 years several talented brains and promising Scholars have been contributed to this area. Some of them are C.F Gauss, G.H Hardy , S. Ramanujan ,A.P Prudnikov , W.W Bell , Yu. A Brychkov and G.E Andrews.

We have the generalized Gaussian hypergeometric function of one variable

$${}_AF_B(a_1,a_2,\dots,a_A;b_1,b_2,\dots b_B;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\dots(a_A)_k z^k}{(b_1)_k(b_2)_k\dots(b_B)_k\, k!} \quad (1)$$

where the parameters b_1 , b_2 , ...,b_B are neither zero nor negative integers and A , B are non negative integers.

The series converges for all finite z if A ≤ B, converges for |z|<1 if $A=B+1$, diverges for all z, z ≠ 0 if A > B+1.

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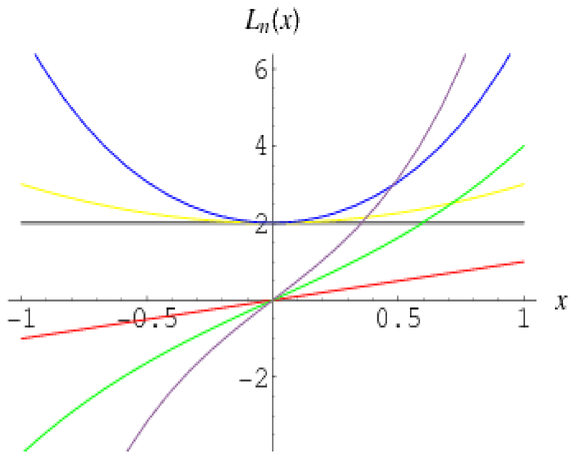
Lucas Polynomials



The Lucas polynomials are the w-polynomials obtained by setting p(x) = x and q(x) = 1 in the Lucas polynomials sequence. It is given explicitly by

$$L_n(x) = 2^{-n}[(x-\sqrt{x^2+4})^n + (x+\sqrt{x^2+4})^n] \quad (2)$$

The first few are

$$\left.\begin{aligned} L_1(x) &= x \\ L_2(x) &= x^2+2 \\ L_3(x) &= x^3+3x \\ L_4(x) &= x^4+4x^2+2 \end{aligned}\right\} \quad (3)$$

Generalized Harmonic Number

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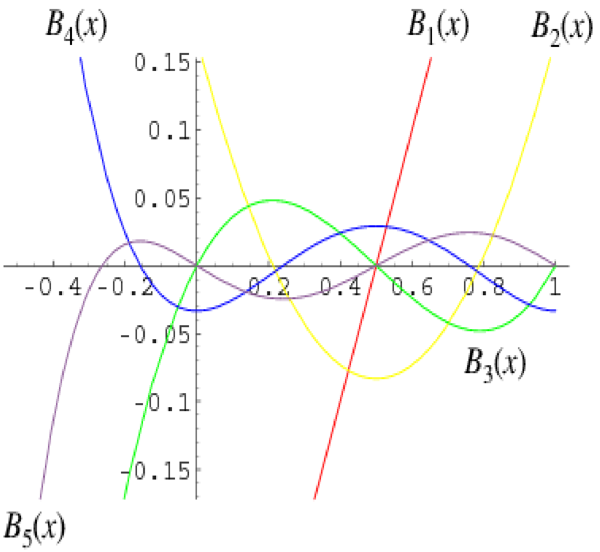
The generalized harmonic number of order *n* of *m* is given by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \tag{4}$$

In the limit of $n \to \infty$ the generalized harmonic number converges to the Riemann zeta function

$$\lim_{n\to\infty} H_n^{(m)} = \varsigma(m) \tag{5}$$

Bernoulli Polynomial



In mathematics, the Bernoulli polynomials occur in the study of many special functions and in particular the Riemann zeta function and Hurwitz zeta function. This is in large part because they are an Appell sequence, i.e. a Sheffer sequence for the ordinary derivative operator. Unlike orthogonal polynomials, the Bernoulli polynomials are remarkable in that the number of crossing of the *x*-axis in the unit interval does not go up as the degree of the polynomials goes up. In the limit of large degree, the Bernoulli polynomials, appropriately scaled, approach the sine and cosine functions.

Explicit formula of Bernoulli polynomials is

$B_n(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} b_{n-k} x^k$, for n≥0, where b_k are the Bernoulli numbers.

The generating function for the Bernoulli polynomials is

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{6}$$

Gegenbauer polynomials

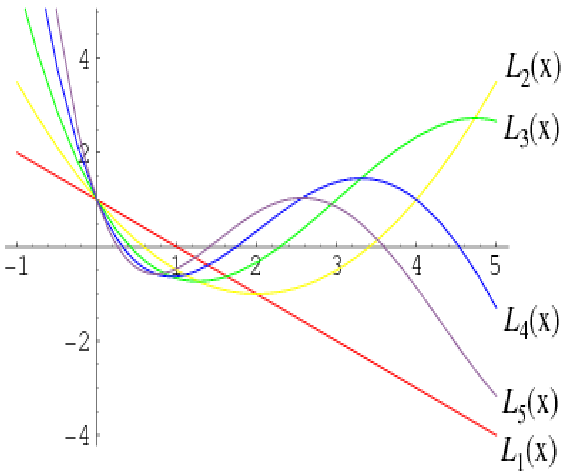
In Mathematics, Gegenbauer polynomials or ultraspherical polynomials are orthogonal polynomials on the interval [-1,1] with respect to the weight function

$(1-x^2)^{\alpha-\frac{1}{2}}$.They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials. They are named after Leopold Gegenbauer.

Explicitly ,

$$C_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!}(2z)^{(n-2k)} \tag{7}$$

Laguerre polynomials



The Laguerre polynomials are solutions $L_n(x)$ to the Laguerre differential equation

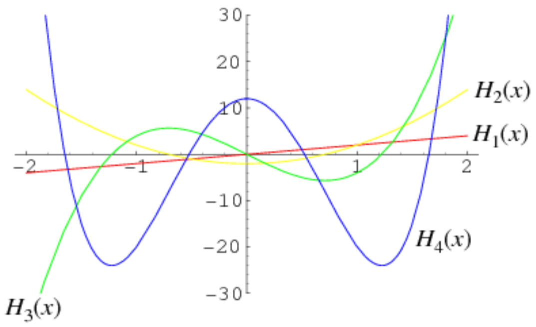
$xy'' + (1-x)y' + \lambda y = 0$, which is a special case of the more general associated Laguerre differential equation, defined by

$xy''+(\nu+1-x)y'+\lambda y=0$, where λ and ν are real numbers with ν =0.

The Laguerre polynomials are given by the sum

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{n!}{k!(n-k)!}x^k \qquad (8)$$

Hermite polynomials



The Hermite polynomials $H_n(x)$ are set of orthogonal polynomials over the domain (-∞,∞) with weighting function e^{-x^2}.

The Hermite polynomials $H_n(x)$ can be defined by the contour integral

$$H_n(z)=\frac{n!}{2\pi i}\oint e^{-r^2+2rz}t^{-n-1}dt,$$

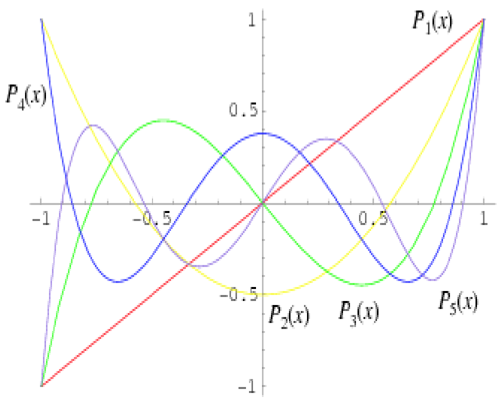
Where the contour incloses the origin and is traversed in a counterclockwise direction

(Arfken 1985, p. 416).

The first few Hermite polynomials are

$$\left.\begin{aligned} H_0(x)&=1\\ H_1(x)&=2x\\ H_2(x)&=4x^2-2\\ H_3(x)&=8x^3-12x\\ H_4(x)&=16x^4-48x^2+12 \end{aligned}\right\} \qquad (9)$$

Legendre function of the first kind



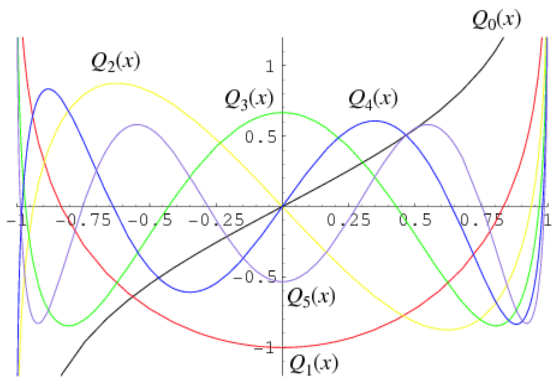
The Legendre polynomials, sometimes called Legendre functions of the first kind, Legendre coefficients, or zonal harmonics (Whittaker and Watson 1990, p. 302), are solutions to the Legendre differential equation. If *l* is an integer, they are polynomials. The Legendre polynomials $P_n(x)$ are illustrated above for $x\in[-1, 1]$ and *n*=1, 2, ..., 5.

The Legendre polynomials $P_n(x)$ can be defined by the contour integral

$$P_n(z)=\frac{1}{2\pi i}\oint(1-2tz+t^2)^{-\frac{1}{2}}t^{-n-1}dt, \qquad (10)$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

Legendre function of the second kind



The second solution $Q_l(x)$ to the Legendre differential equation. The Legendre functions of the second kind

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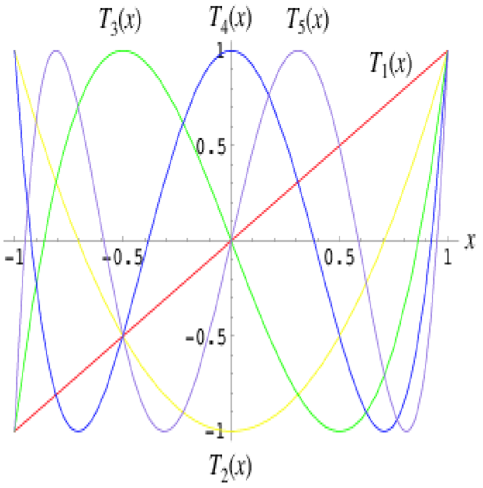
satisfy the same recurrence relation as the Legendre polynomials.

The first few are

$$\left.\begin{aligned}
Q_0(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\
Q_1(x) &= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1 \\
Q_2(x) &= \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{3x}{2}
\end{aligned}\right\} \quad (11)$$

Chebyshev polynomial of the first kind



The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_n(x)$. They are used as an approximation to a least squares fit, and are a special case of the Gegenbauer polynomial with $\alpha = 0$. They are also intimately connected with trigonometric multiple-angle formulas.

The Chebyshev polynomial of the first kind $T_n(z)$ can be defined by the contour integral

$$T_n(z) = \frac{1}{4\pi i}\oint \frac{(1-t^2)t^{-n-1}}{1-2tz+t^2}dt \,, \qquad (12)$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

The first few Chebyshev polynomials of the first kind are

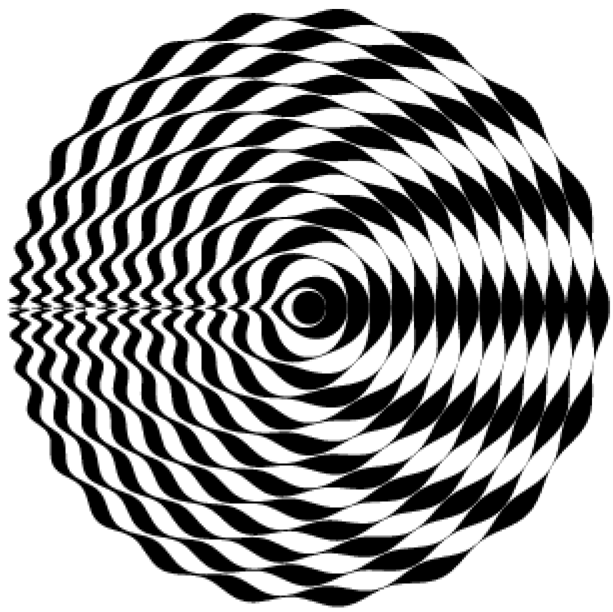
$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

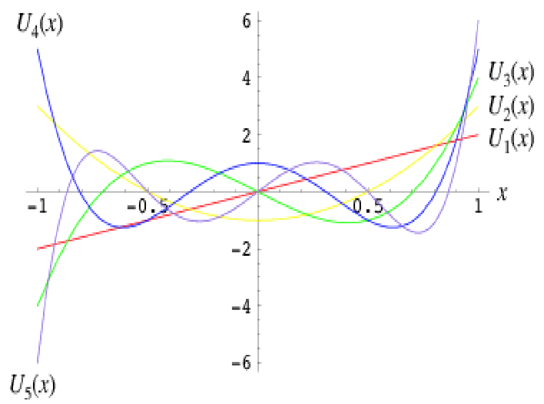
A beautiful plot can be obtained by plotting $T_n(x)$ radially, increasing the radius for each value of n, and filling in the areas between the curves (Trott 1999, pp. 10 and 84).



The Chebyshev polynomials of the first kind are defined through the identity

$T_n(\cos\theta) = \cos n\theta$.

Chebyshev polynomial of the second kind



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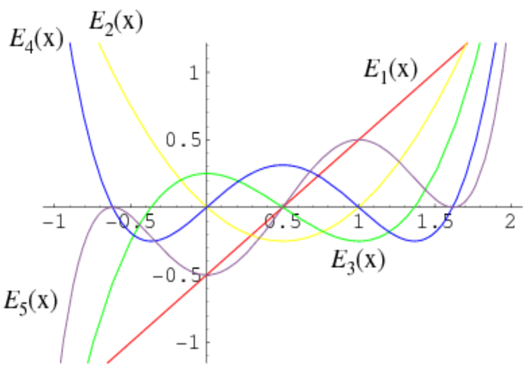
A modified set of Chebyshev polynomials defined by a slightly different generating function. They arise in the development of four-dimensional spherical harmonics in angular momentum theory. They are a special case of the Gegenbauer polynomial with $\alpha = 1$. They are also intimately connected with trigonometric multiple-angle formulas.

The first few Chebyshev polynomials of the second kind are

$$\left.\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \end{aligned}\right\} \qquad (13)$$

Euler polynomial



The Euler polynomial $E_n(x)$ is given by the Appell sequence with

$$g(t) = \frac{1}{2}(e^t + 1),$$

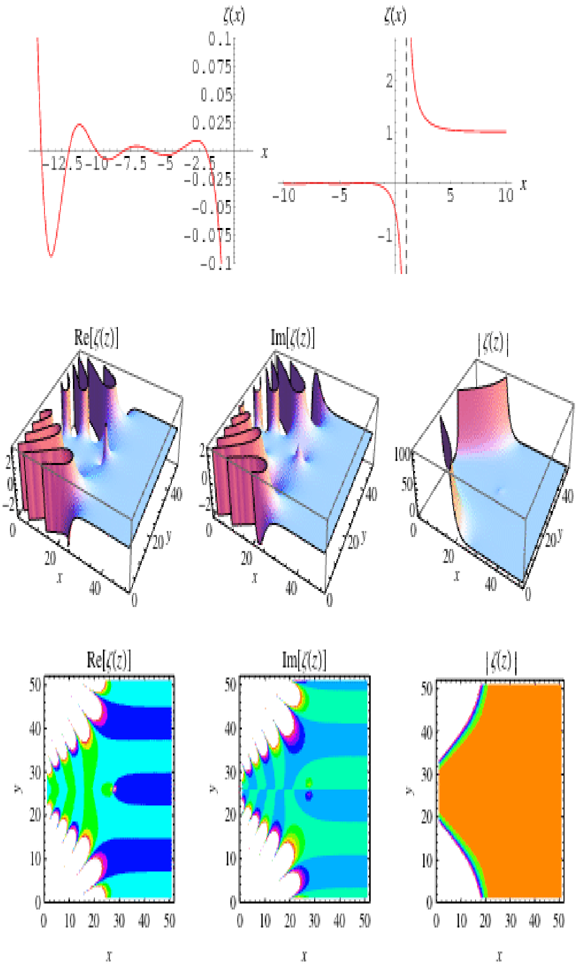
giving the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \qquad (14)$$

The first few Euler polynomials are

$$\left.\begin{aligned} E_0(x) &= 1 \\ E_1(x) &= x - \frac{1}{2} \\ E_2(x) &= x^2 - x \\ E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4} \end{aligned}\right\} \qquad (15)$$

Generalized Riemann zeta function



The Riemann zeta function is an extremely important special function of mathematics and physics that arises in definite integration and is intimately related with very deep results surrounding the prime number theorem. While many of the properties of this function have been investigated, there remain important fundamental conjectures (most notably the Riemann hypothesis) that remain unproved to this day. The Riemann zeta function $\zeta(s)$ is defined over the complex

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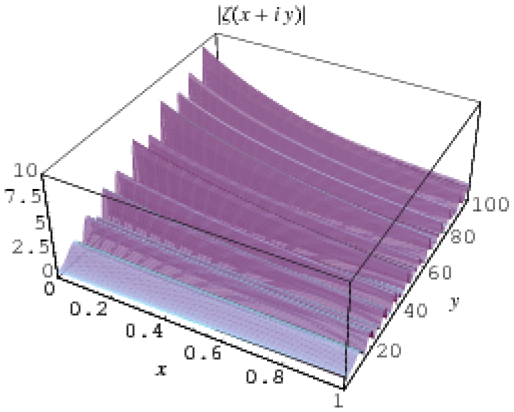
plane for one complex variable, which is conventionally denoted s (instead of the usual z) in deference to the notation used by Riemann in his 1859 paper that founded the study of this function (Riemann 1859).



The plot above shows the "ridges" of $|\zeta(x+iy)|$ for $0<x<1$ and $1<y<100$. The fact that the ridges appear to decrease monotonically for $0 \le x \le 1/2$ is not a coincidence since it turns out that monotonic decrease implies the Riemann hypothesis (Zvengrowski and Saidak 2003; Borwein and Bailey 2003, pp. 95-96).

On the real line with $x>1$, the Riemann zeta function can be defined by the integral

$\varsigma(x) \equiv \frac{1}{\Gamma(x)}\int_0^\infty \frac{u^{x-1}}{e^u-1}du$, where $\Gamma(x)$ is the gamma function.

Complex infinity

Complex infinity is an infinite number in the complex plane whose complex argument is unknown or undefined. Complex infinity may be returned by *Mathematica*, where it is represented symbolically by ComplexInfinity. The Wolfram Functions Site uses the notation $\tilde{\infty}$ to represent complex infinity.

2. **MAIN RESULTS**

$$\int L_1(x)(1+e^x)^{\frac{1}{n}}dx = n(e^{-x}+1)^{-\frac{1}{n}}(e^x+1)^{\frac{1}{n}}[x\{{}_2F_1(-\frac{1}{n},-\frac{1}{n};\frac{n-1}{n};-e^{-x})\}$$
$$-n\{{}_3F_2(-\frac{1}{n},-\frac{1}{n},-\frac{1}{n};\frac{n-1}{n},\frac{n-1}{n};-e^{-x})\}]+C \quad (17)$$

Here $L_1(x)$ is **Lucas Polynomials.**

$$\int H_1^{(x)}\sin^{\frac{1}{n}}(x)dx = -\cos(x)\sin^{\frac{n+1}{2}}(x)\sin^2(x)^{\frac{1}{2}(-\frac{1}{n}-1)}{}_2F_1(\frac{1}{2},\frac{1}{2}(1-\frac{1}{n});\frac{3}{2};\cos^2(x))+C \quad (18)$$

Here $H_1^{(x)}$ is **Generalized Harmonic number**.

$$\int \zeta(1,x)\sin^{\frac{1}{n}}(x)dx = \tilde{\infty}+C \quad (19)$$

Here ζ(x) is **Generalized Riemann zeta function.**

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$$\int H_1(x)\sin^{\frac{1}{n}}(x)dx=\sin^{\frac{1}{n}+1}(x)[\frac{2x\cos(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))\}}{\frac{1}{n}+1} -\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\{{}_3F_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\}]+C \tag{20}$$

Here H_1(x) is **Hermite polynomials**.

$$\int U_1(x)\sin^{\frac{1}{n}}(x)dx=\sin^{\frac{1}{n}+1}(x)[\frac{2x\cos(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))\}}{\frac{1}{n}+1} -\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\{{}_3F_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\}]+C \tag{21}$$

Here U_1(x) is **Chebyshev polynomial of the second kind.**

$$\int T_1(x)\sin^{\frac{1}{n}}(x)dx=\frac{1}{2}\sin^{\frac{1}{n}+1}(x)[\frac{2x\cos(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))\}}{\frac{1}{n}+1} -\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\{{}_3F_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\}]+C \tag{22}$$

Here T_1(x) is **Chebyshev polynomial of the first kind.**

$$\int P_1(x)\sin^{\frac{1}{n}}(x)dx=\frac{1}{2}\sin^{\frac{1}{n}+1}(x)[\frac{2x\cos(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))\}}{\frac{1}{n}+1} -\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\{{}_3F_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\}]+C \tag{23}$$

Here P_1(x) is **Legendre function of the first kind.**

$$\int H_1(x)\sec^{\frac{1}{n}}(x)dx=\sin(x)\cos^2(x)^{\frac{n+1}{2n}}\sec^{1+\frac{1}{n}}(x)\{{}_2F_1(\frac{1}{2},\frac{1}{2}(1+\frac{1}{n});\frac{3}{2};\sin^2(x))\}+C \tag{24}$$

$$\int H_1^{(x)}\cos ec^{\frac{1}{n}}(x)dx=-\cos(x)\sin^2(x)^{\frac{1}{2}(\frac{1}{n}-1)}\cos ec^{\frac{1}{n}-1}(x){}_2F_1(\frac{1}{2},\frac{1}{2}(1+\frac{1}{n});\frac{3}{2};\cos^2(x))+C \tag{25}$$

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$$\int F_1(x)\sin^{\frac{1}{n}}(x)dx = \frac{1}{2}\sin^{\frac{1}{n}+1}(x)\Big[\frac{2x\cos(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))\}}{\frac{1}{n}+1}$$
$$-\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\{{}_3\tilde{F}_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\}\Big]+C \qquad (26)$$

$$\int L_1(x)\sin^{\frac{1}{n}}(x)dx = \frac{1}{2}\sin^{\frac{1}{n}+1}(x)\Big[\frac{2x\cos(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))\}}{\frac{1}{n}+1}$$
$$-\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\{{}_3\tilde{F}_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\}\Big]+C \qquad (27)$$

$$\int C_1(x)\sin^{\frac{1}{n}}(x)dx = \sin^{\frac{1}{n}+1}(x)\Big[\frac{2x\cos(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))\}}{\frac{1}{n}+1}$$
$$-\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\{{}_3\tilde{F}_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\}\Big]+C \qquad (28)$$

Here C_1 (x) is **Gegenbauer polynomials.**

$$\int B_1(x)\cos^{\frac{1}{n}}(x)dx = \frac{1}{2}\cos^{\frac{1}{n}+1}(x)\Big[\sqrt{\pi}(-2^{-\frac{1}{n}-1})\Gamma(1+\frac{1}{n})\cos(x)$$
$$*{}_3\tilde{F}_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\cos^2(x))$$
$$-\frac{2x\sin(x)\{{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\cos^2(x))\}}{\frac{1}{n}+1}$$
$$+\frac{\sin(x)\cos^{\frac{1}{n}+1}(x)\{{}_2F_1(\frac{1}{2},\frac{1}{2}(1+\frac{1}{n});\frac{1}{2}(3+\frac{1}{n});\cos^2(x)}{2(1+\frac{1}{n})\sqrt{\sin^2(x)}}+C \qquad (29)$$

Here B_1 (x) is **Bernoulli Polynomial.**

$$.\int H_1^{(x)}\cos^{\frac{1}{n}}(x)dx = \frac{\sin(x)\cos^{\frac{1}{n}+1}(x)\,{}_2F_1(\frac{1}{2},\frac{1}{2}(1+\frac{1}{n});\frac{1}{2}(3+\frac{1}{n});\cos^2(x)}{(1+\frac{1}{n})\sqrt{\sin^2(x)}}+C \qquad (30)$$

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$$\begin{aligned}\int L_1(x)\sin^{\frac{1}{n}}(x)dx = &-\frac{1}{2}\sin^{\frac{1}{n}+1}(x)[-\sqrt{\pi}\,2^{-\frac{1}{n}-1}\Gamma(1+\frac{1}{n})\sin(x)\\ &*{}_3\tilde{F}_2(1,1+\frac{1}{2n},1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n}),2+\frac{1}{2n};\sin^2(x))\\ &+2\cos(x)\sin^2(x)^{-\frac{n+1}{2n}}{}_2F_1(\frac{1}{2},\frac{n-1}{2n};\frac{3}{2};\cos^2(x))\\ &+\frac{2nx\cos(x)\,{}_2F_1(1,1+\frac{1}{2n};\frac{1}{2}(3+\frac{1}{n});\sin^2(x))}{n+1}]+C\end{aligned} \tag{31}$$

Here $L_1(x)$ is **Laguerre polynomials.**

$$\begin{aligned}\int B_1(x)\operatorname{cosec}^{\frac{1}{n}}(x)dx = &\frac{1}{2}\operatorname{cosec}^{\frac{1}{n}+1}(x)[\sqrt{\pi}(-2^{\frac{1}{n}-1})\Gamma(1-\frac{1}{n})\sin(x)\\ &*{}_3\tilde{F}_2(1,1-\frac{1}{2n},1-\frac{1}{2n};\frac{1}{2}(3-\frac{1}{n}),2-\frac{1}{2n};\sin^2(x))\\ &+\cos(x)\sin^2(x)^{\frac{1}{2}(\frac{1}{n}-1)}{}_2F_1(\frac{1}{2},\frac{1}{2}(1+\frac{1}{n});\frac{3}{2};\cos^2(x))\\ &+\frac{2nx\cos(x)\,{}_2F_1(1,1-\frac{1}{2n};\frac{1}{2}(3-\frac{1}{n});\sin^2(x))}{n-1}]+C\end{aligned} \tag{32}$$

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