Reverse degree-based topological indices of Unitary Cayley graphs

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Abstract -*The unitary Cayley graph X_n has vertex set Z_n = {0, 1, 2, . . n – 1}. Vertices a, b are adjacent, if $gcd(a - b, n) = 1$. The degree $deg\ (u)$ of a vertex u in a graph G is the number of vertices adjacent to u. In this paper, we obtain the values of the reverse degree based topological indices of unitary Cayley graph X_n.*

Key Words: Unitary Cayley graphs, reverse degree based topological indices, reverse Zagreb index, hyper-Zagreb indices.

Mathematics Subject Classification: 05C09, 05C25, 05C50

1. PRELIMINARIES:

A graph $G = (V, E)$ in which $V\ (G)$ is the set of vertices and $E\ (G)$ is the set of edges. An edge $\{u, v\}$ is an unordered pair of distinct vertices; the vertices u and v are the endpoints of the edge. The vertices of a graph can be pictured as dots, and the edges as line segments whose endpoints are vertices. An edge e is said to be incident at a vertex v if v is an endpoint of e. Two vertices are said to be adjacent if there is an edge between them. Adjacent vertices are sometimes referred to as neighbors. The degree $deg\ (u)$ of a vertex u in a graph G is the number of vertices adjacent to u. A graph is simple if there is at most one edge between any two vertices, and the endpoints of each edge are distinct.

Researchers have introduced many regular graphs using Cayley graphs with more number of vertices for a given diameter and for a given number of edges per vertex that were studied previously. This lead to the construction of larger networks, while meeting design criteria of a fixed number of nearest neighbors and a fixed maximum communication time between arbitrary vertices. An immediate example of Cayley graph is a unitary Cayley graph. More information about Cayley graphs can be found in the books on algebraic graph theory by Biggs [1] and by Godsil and Royle [2]. Unitary Cayley graphs are highly symmetric. They have some remarkable properties connecting graph theory and number theory.

Topological indices are mathematical descriptors of the structure of a molecule that can be used to predict its properties. They are derived from the graph theory, which describes the topology of a molecule and its connectivity. The authors have simultaneously computed the first and second Reverse Zagreb indices, reverse hyper-Zagreb indices, and their polynomials. This research also derives mathematical closed-form formulas for some of its fundamental degree-based molecular descriptors.

All graphs discussed in this paper are simple graphs. The unitary Cayley graph X_n has vertex set Z_n $= \{0, 1, 2, . . n - 1\}$ where the vertices a, b are adjacent, if $gcd\ (a - b, n) = 1$. If we represent the elements of Z_n by the integers 0, 1, . . . , $n - 1$, then it is well known that X_n has vertex set $V(X_n) = Z_n = \{0, 1, \ldots . . n - 1\}$ and edge set ( ) = $\{\{a, b\}: a, b \in Z_n, \gcd(a - b, n) = 1\}$.

In section 2 we study some new indices of unitary Cayley graphs. A numerical parameter mathematically derived from the graph structure is called as a topological index or graph index. Topological indices uniquely characterize molecular topology and is extensively applied in chemical graph theory. Application of ideas from one scientific field to another one often gives a new view on the problems. Mostly we can find the applications of graph indices [3] towards various fields of Science and Technology in [4, 5]. One can find that graph indices are extensively studied in [6, 7, 8]. The status, denoted by (u), of a vertex u in G is the sum of distances of all other vertices from u in G.

The paper proceeds further on the successful consideration of degree based indices.

The number of edges that a vertex is associated with determines its degree. A graph's maximum vertex degree is represented by the symbol $\Delta(G)$.Kulli introduces the concept

of reverse vertex degree with the formula as

$$Cv = \Delta(G) - deg(V) + 1$$

Where Cv is denoted by the reverse degree, $\Delta(G)$ is the maximum degree and $dg(V)$ shows the minimum vertices adjacent to v [**9**, **10**].

Gutman (1972) defined and formulated the first and second Zagreb indices [**11**]. We defined the $CM1(G)$ and $CM2(G)$ reverse Zagreb index.

$$CM1(G) = \sum uv\epsilon E(G)(cu + cv)$$

$$CM2(G) = \sum uv\epsilon E(G)(cu \times cv)$$

Ivan Gutman introduced the first and second hyper-Zagreb indice [**11, 12, 13**]. We defined the $HCM1(G)$ and $HCM2(G)$ reverse Zagreb index.

$$HCM1(G) = \sum uv\epsilon E(G)(cu + cv)^2$$

$$HCM2(G) = \sum uv\epsilon E(G)(cu \times cv)^2$$

The first and second Zagreb polynomials were introduced by Ivan Gutman [**14**]. We defined the $CM1(G.x)$ and $CM2(G.x)$ reverse Zagreb index.

$$CM1(G.x) = \sum uv\epsilon E(G)x(cu + cv)$$

$$CM2(G.x) = \sum uv\epsilon E(G)x(cu \times cv)$$

The same paper first and second hyper-Zagreb indices [**12**]. We defined the $HCM1(G.x)$ and $HCM2(G.x)$ reverse Zagreb index.

2. METHODOLOGY:

The unitary Cayley graph has vertex set $Z_n = \{0,1,2,\ldots n-1\}$ vertices a and b are adjacent, if $gcd(a-b,n) = 1$. The status $\sigma(u)$ of a vertex u in a graph, G is the sum of distances of all other vertices from u in G. In this paper, we obtain the values of the Reverse degree based topological indices for unitary Cayley graph X_n. Different values of Reverse degree based topological indices have been obtained for unitary Cayley graphs for different numbers of vertices (odd and even number of vertices).

3. RESULTS AND DISCUSSIONS:

The values of Reverse degree based topological indices have been extensively studied by various researchers recently and most of the research has been done for different types of graphs in chemical graph theory. The present work reveals that the unitary Cayley graph is one of the graphs that is extensively studied in both the areas of group theory and graph theory. We derive the values of Reverse degree based topological indices that are dependent on the degree of the vertices of the unitary Cayley graph.

Lemma 1(15): Let X_n be a unitary Cayley graph where $n = p$ is a prime number, then $Xn = Kp$ is a complete graph on p vertices.

Lemma 2(15): Let X_n be a unitary Cayley graphs for n as any even number. In specific when n = 6, X_n becomes a cycle.

Theorem 3: The reverse vertex degree of a unitary Cayley graph X_n where $n = p$ is a prime number is 1.

Proof: X_n is a unitary Cayley graph where$n$=p is a prime number, then X_n=K_p is a complete graph on p vertices. In a complete graph, we have $n(n-1)/2$ edges. Then

$$Cv = \Delta(G) - d_{X_n}(V) + 1 = (n-1) - (n-1) + 1 = 1.$$

Observations:

3.1: From (16), the $CM1(X_n)$ and $CM2(X_n)$ reverse Zagreb index. of a unitary Cayley graph X_n where n=p is a prime number is

$$CM1(X_n) = \sum uv\epsilon E(X_n)(cu + cv) = \frac{n(n-1)}{2}(2) = \mathrm{n}(n-1)$$
$$CM2(X_n) = \sum uv\epsilon E(X_n)(cu \times cv) = \frac{n(n-1)}{2}(1) = \frac{n(n-1)}{2}$$

3.2: From (16), the $HCM1(X_n)$ and $HCM2(X_n)$ reverse Zagreb index of a unitary Cayley graph X_n where n=p is a prime number is

$$HCM1(X_n) = \sum uv\epsilon E(X_n)(cu + cv)^2 = \frac{n(n-1)}{2}(4) = 2n(n-1)$$

$$HCM2(X_n) = \sum uv\epsilon E(X_n)(cu \times cv)^2 = \frac{n(n-1)}{2}(1) = \frac{n(n-1)}{2}$$

3.3: From (16), the $CM1(X_n.x)$ and $CM2(X_n.x)$ reverse Zagreb index of a unitary Cayley graph X_n where n=p is a prime number is

$$CM1(X_n.x) = \sum uv\epsilon E(X_n)x(cu + cv) = \frac{n(n-1)}{2}x(2) = xn(n-1)$$

$$CM2(X_n.x) = \sum uv\epsilon E(X_n)x(cu \times cv) = \frac{n(n-1)}{2}x(1) = \frac{n(n-1)}{2}x$$

We continue our study of reverse degree based indices of unitary Cayley graphs for n as any even number. Specifically when n=6, becomes a cycle.

Theorem 4: The reverse vertex degree of a unitary Cayley graph X_n where n=6 is an even number is 1.

Proof: When n=6, is a cycle, then it has n edges and every vertex has degree 2. Then

$$Cv = \Delta(G) - d_{X_n}(V) + 1 = 2 - 2 + 1 = 1.$$

Observations:

4.1: From (16), the $CM1(X_n)$ and $CM2(X_n)$ reverse Zagreb index. of a unitary Cayley graph X_n where n=p is a prime number is

$$CM1(X_n) = \sum uv\epsilon E(X_n)(cu + cv) = n(2) = 2n$$

$$CM2(X_n) = \sum uv\epsilon E(X_n)(cu \times cv) = n(1) = n$$

4.2: From (16), the $HCM1(X_n)$ and $HCM2(X_n)$ reverse Zagreb index of a unitary Cayley graph X_n where$n$=p is a prime number is

$$HCM1(X_n) = \sum uv\epsilon E(X_n)(cu + cv)^2 = n(4) = 4n$$

$$HCM2(X_n) = \sum uv\epsilon E(X_n)(cu \times cv)^2 = n(1) = n$$

4.3: From (16), the $CM1(X_n.x)$ and $CM2(X_n.x)$ reverse Zagreb index of a unitary Cayley graph X_n where n=p is a prime number is

$$CM1(X_n.x) = \sum uv\epsilon E(X_n)x(cu + cv) = nx(2) = 2nx$$

$$CM2(X_n.x) = \sum uv\epsilon E(X_n)x(cu \times cv) = nx(1) = nx.$$

Further, we study the reverse degree based indices of unitary Cayley graphs for n as any even number such that n = 2p where p is a prime and $p \neq 2,3$.

Theorem 5: The reverse vertex degree of a unitary Cayley graph X_n for n as any even number such that n = 2p where p is a prime and $p \neq 2,3$ is 1.

Proof: This type of graph has $2n$ edges, every vertex has (u) = n – 1. Then

$$Cv = \Delta(G) - d_{X_n}(V) + 1 = (n-1) - (n-1) + 1 = 1.$$

Observations:

5.1: From (16), the $CM1(X_n)$ and $CM2(X_n)$ reverse Zagreb index of a unitary Cayley graph X_n for n as any even number such that n = 2p where p is a prime and $p \neq 2,3$ is

$$CM1(X_n) = \sum uv\epsilon E(X_n)(cu + cv) = 2n(2) = 4n$$

$$CM2(X_n) = \sum uv\epsilon E(X_n)(cu \times cv) = 2n(1) = 2n$$

5.2: From (16), the $HCM1(X_n)$ and $HCM2(X_n)$ reverse Zagreb index of a unitary Cayley graph for n as any even number such that n = 2p where p is a prime and $p \neq 2,3$ is

$$HCM1(X_n) = \sum uv\epsilon E(X_n)(cu + cv)^2 = 2n(4) = 8n$$

$$HCM2(X_n) = \sum uv\epsilon E(X_n)(cu \times cv)^2 = 2n(1) = 2n$$

5.3: From (16), the $CM1(X_n.x)$ and $CM2(X_n.x)$ reverse Zagreb index of a unitary Cayley graph X_n for n as any even number such that n = 2p where p is a prime and $p \neq 2,3$ is

$$CM1(X_n.x) = \sum uv\epsilon E(X_n)x(cu + cv) = 2nx(2) = 4nx$$

$$CM2(X_n.x) = \sum uv\epsilon E(X_n)x(cu \times cv) = 2nx(1) = 2nx.$$

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