Study on Non-Negative Integer Solutions of Exponential Diophantine Equation $512^x + 1728^y = z^3$ and $271^x + 9^y = z^3$

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Abstract – *The aim of the present paper is to demonstrate the problem of existence of the solution of exponential non-linear Diophantine equation as there are no general methods to find solution with natural number. This is an attempt to find numerical solutions (if any) of the equations $512^x + 1728^y = z^3$, and the $271^x + 9^y = z^3$, where (x, y, z) are non-negative integers.*

***Keywords*: Exponential Diophantine equation, Number Theory, Non-Negative Integers Solution**
Mathematics Subject Classification (ASM): 11D61, 11D79

1. INTRODUCTION

The theory of Number is an elegant branch of mathematics that primarily concerned with the study of non-negative integers, or counting numbers, and their properties as well as the solvability of equations in whole numbers. It has a veritably long and different history, and some of the topmost mathematicians of all time, similar as Euclid, Euler, and Gauss, have made significant benefactions to it. Hardy and Wright, [9] discussed a great diversity of different topics of theoretical number theory and found a remarkable selection of arithmetic problems treated with consummate clarity and distinction. Burton [7] suggested the study of Elementary & classical number theory and to impart some of the historical background in which the subject evolved. Niven et al. [11] have discussed the introduction to the theory of numbers and expand the binomial theorem, calculation methods for numerical and a public key cryptography section. Contains an outstanding set of problems. Baker, [1] and [2] provided comprehensive initiation to all the major branches of number theory including elements of cryptography and primality testing, an account of number fields in the arithmetic of elliptic curves. The particular type of Exponential Diophantine equation is analyzed and generalized by the method of Catalan's conjecture, its primary Cyclotomic units, and proof was given by Mihailescu [15].

Algebraic equations with non-negative integer amounts having integer solutions are Diophantine equations. For finding the solution to these equations, there's no universal manner available yet, so the investigators are keenly interested in developing new techniques for unravelling these equations. While handling any cognate equation, three issues arise, that's whether the problem is resolvable or not; if resolvable, possible number of solutions and finally to find the complete results. Diophantine equations are frequently used in the field of Abstract algebra, Coordinate geometry, Group theory, Linear algebra, Trigonometry, Cryptography and asunder as well as we can define the number of rational points on a circle. In investigations on Diophantine equations of steps, two significant successes were scored only in the 20th century. It was proved by A. Thue [4]. Diophantus from Alexandria such equations are vociferated Diophantine equations. Mordell [12] studied the Diophantine equations. Acu [6] has studied the elementary solutions for the Diophantine equation of $a^x + b^y = c^z$. Suvarnamani et al. [3] analyzed two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. Sroysang [5] obtained the solutions for Diophantine Equations $2^x + 3^y = z^2$. Cohen [10] studied the Number Theory and also gave its tools he also dealt with many aspects of number theory, mainly the central theme being the solution of Diophantine equations. Cipu et al. [13] have revealed the number of extensions for a fixed Diophantine triple. Burshtein [14], considered the general equation of three consecutive prime integers of the form $p^x + (p + 1)^y + (p + 2)^z = M^3$, where M represents a positive integer and p represents prime with $p \geq 2$, $x, y \geq 1$, & $z \leq 2$, also he determined all solutions of the above exponential Diophantine equations. Janaki and Shankari [8] have discussed various implementable ways to tackle multivariable and multi-degree Diophantine problems and obtain the solution of these exponential Diophantine equations.

The paper is organized as follows. Section 2 presents the Preliminary work of the paper by using Lemma and theorems. Section 3 and 4, presents the working strategy to solve the main exponential Diophantine equations of this paper. The conclusions about the obtained solutions are contain in section 5. The rest of the paper listed the related work as references.

2. Preliminary

2.1 Lemma: (Miheailescu's Theorem) [15] The Dio-phantine equation $a^x - b^y = 1$ has the unique solution (a, b, x, y) = (3, 2, 2, 3), where a, b, x and y are integers with min {a, b, x, y} > 1.

2.2 Lemma: There are no solutions in integer x, y, z with x, y, z > 0, of $x^n + y^n = z^n$ when $n \geq 3$, also known as Fermat's Last Theorem, was proved by Wiles [16].

3. Main work for Equation 1: $512^x + 1728^y = z^3$

Theorem 3: The Diophantine $512^x + 1728^y = z^3$, has no solution in non-negative integers, where x, y, z > 0.

Proof: let we find the trivial solution for

$$512^x + 1728^y = z^3 \tag{3}$$

Let x = 0

$$1 + 1728^y = z^3, \text{ then } z^3 > 1 \text{ or } z > 1$$

or $z^3 - 1728^y = 1$, by Lemma 2.1, y can only take the value 0 and1, for y = 0 and 1, the equation $1 + 1728^y = z^3$ gives z has no integer value.

Now let y = 0

$512^x + 1 = z^3$, also here $z^3 > 1$ or $z > 1$, then equation $z^3 - 512^x = 1$, by Lemma 2.1, says, x can take the value less than 2 in integers that are x = 0, 1. for x = 0 and 1 equation $512^x + 1 = z^3$ implying $z^3 = 2$ and 513 respectively, it gives z has not a positive integer value. Hence, the equation $512^x + 1728^y = z^3$ has no trivial solution.

Now find the nontrivial solution for equation, (3)

or $512^x + 1728^y = z^3$

can be rewrite of the form

$$8^{3x} + 12^{3y} = z^3 \tag{3.1}$$

$\Longrightarrow$ $2|z$

z is even because LHS is even, if z is odd then z^3 is also odd Dividing equation (3.1) by 8 we get

$$8^{3x-1} + 12^{3y-3} \cdot 6^3 = (z/2)^3$$

In general, if we divide (3.1) by 8, *i* times then

$$8^{3x-i} + 12^{3y-\begin{cases}\frac{3i}{2} & \text{if } i \text{ is even}\\ \frac{3i+3}{2} & \text{if } i \text{ is odd}\end{cases}} \cdot 6^{\begin{cases}0 & \text{if } i \text{ is even}\\ 3 & \text{if } i \text{ is odd}\end{cases}} \cdot 3^{\begin{cases}\frac{3i}{2} & \text{if } i \text{ is even}\\ \frac{3i-3}{2} & \text{if } i \text{ is odd}\end{cases}} = \left(\frac{Z}{2^i}\right)^3 \quad \text{for } i>1 \tag{3.2}$$

We cannot divide by 8 infinitely many times. In equation (3.2) when i = min {3x, 2y, L} where L is the logarithm of highest power of 2 dividing z on base 2, it cannot be divided by 8. We assume in (3.1) that the term is reduced if it stops the equation to divide by 8 and can be an integer, first.

Case 3.1: if 12^{3y} reduces before or with z^3 or 8^{3x}

After keep on dividing by 8, 2y times, then

$$8^{3x-2y} + 3^{3y} = (z/2^{2y})^3$$

$$3x-2y \geq 0$$

$$8^{3x-2y}+3^{3y} = t^3 \qquad [t = z/2^{2y}]$$

$$(2^{3x-2y})^3+(3^y)^3 = t^3$$

Let $a = 2^{3x-2y}$, $b = 3^{3y}$

$a^3 + b^3 = t^3$ which takes form of $x^3 + y^3 = z^3$

and has no solution due to Lemma 2.2.

if 12^{3y} will not reduce first go to the next case.

Case 3.2: let 8^{3x} reduces first before z^3

$$1 + 12^{3y-9x} \cdot 6^{9x} = (z/8^x)^3$$

Put, m = 3x

$$1+(12^{y-m} \cdot 6^m)^3 = (z/2^m)^3 \tag{3.2.1}$$

$$z/2^m > 1$$

as $z/2^m$ is even

As 12^{3y} has not reduced, so y-m > 0 or y > m

$12^{y-m} \cdot 6^m > 1$ and by Catalan conjecture Lemma 2.1, we have no solution for this case.

Case 3.3: if z^3 reduces first to let say k^3 and before 8^{3x}

Let $z = 2^m k$,

Dividing by 8 up to m times of equation (3.1)

$$8^{3x-m} + 12^{3y-3m} \cdot 6^{3m} = k^3$$

And it can be clearly seen that parity of LHS and RHS are different

Case 3.4: if z^3 and 8^{3x} reduce both

After reducing, we get

$$1 + (6^{3x} \cdot 12^{y-3x})^3 = k^3, \qquad \text{where } z = 2^{3x}k$$

(8 is divided 3x times)

If $n = 6^{3x} \cdot 12^{3y-3x}$, $1+n^3 = k^3$

Now, n > 1, k > 0

If k is 1 then n is 0 which is not possible so k > 1 which has no solution by Lemma 2.1.

Hence, the equation $512^x + 1728^y = z^3$ has no solution in non-negative integers.

4. Main work for Equation 2: $271^x + 9^y = z^3$

Theorem 4: Let *x*, *y*, *z* be non-negative integers then the equation $271^x + 9^y = z^3$ has a solution (1, 3, 10).

Proof: Now initially check the trivial solution for the equation

$$271^x + 9^y = z^3 \tag{4}$$

Let x = 0,

$$1 + 9^y = z^3 \qquad \text{Since } z^3 > 1, \text{ so } z > 1$$

or $z^3 - 9^y = 1$,

By the Lemma 2.1, y can take the value only less than 2, that is, y = 0, 1. if y = 0, the equation $1 + 9^y = z^3$ gives $z^3 = 2$, which shows z is not an integer value. Similarly, for y = 1, then $1 + 9^1 = z^3$ or $z^3 = 10$ which gives again, z is not an integer value.

Now let y = 0

$271^x + 1 = z^3$ Since $z^3 > 1$, $z > 1$

or $z^3 - 271^x = 1$

By Lemma 2.1, if an equation of the form $a^x - b^y = 1$ then (a, b, x, y) can only take the possible solution is (3, 2, 2, 3) for min {a, b, x, y} > 1. So, x can take the value less than 2 in integers, that are

x = 0, 1.

for x = 0 and 1, the equation $271^x + 1 = z^3$, imply $z^3 = 2$ and 272 respectively for both cases shows z is not a positive integer. Hence the equation $271^x + 9^y = z^3$ has no trivial solution.

Now check the nontrivial solution for equation (4). x, y and z are the non-negative integers and the equation as given below-

Let, $271^x + 9^y = z^3$

z is even so, $z^3 \equiv 0 \pmod 8$,

$9^y \equiv 1 \pmod 8$,

$271 \equiv -1 \pmod 8$, hence x is odd.

Now for finding y, consider it under modulo 3 then there are following cases arise:

Case 4.1: Let y = 3k, then equation (4) becomes

$271^x + 9^{3k} = z^3$

$271^x = z^3 - (9^k)^3$

$271^x = (z - 9^k)(z^2 + 9^k z + 9^{2k})$

Let x = u+v. for factoring 271^x to get values of $z - 9^k$ and $z^2 + 9^k z + 9^{2k}$.

$$271^u = z - 9^k \quad (4.1.1)$$

$$271^v = z^2 + 9^k z + 9^{2k} \quad (4.1.2)$$

From (4.1.1), $z = 271^u + 9^k$

substituting this in equation (4.1.2)

$271^v = (271^u + 9^k)^2 + 9^k(271^u + 9^k) + 9^{2k}$

$= 271^{2u} + 3 \cdot 9^k(271^u + 9^k) = 271^{2u} + 3 \cdot 9^k z$

$3^{2k+1} \cdot z = 271^v - 271^{2u} = 271^{2u}(271^{v-2u} - 1)$

Obviously, v > 2u and $271 \nmid$ LHS hence u=0 and v=x

$\because$ x = u+v;

Then, $3^{2k+1} \cdot z + 1 = 271^x$

or $z^3 - 3^{2k+1} \cdot z - (3^{6k} + 1) = 0$

which gives, $z = (3^{2k} + 1), (-3^{2k} - 1 \pm \sqrt{(-4 \cdot 3^{2k+1})})$

Now, consider only real value of $z = 3^{2k} + 1$, then

$3^{2k+1}(3^{2k} + 1) + 1 = 271^x$

$3^{4k+1} + 3^{2k+1} = 271^x - 1^x$

$3^{4k-2} + 3^{2k-2} = 10(271^{x-1} + 271^{x-2} + \ldots + 271^2 + 271 + 1)$

If k = 1, then $3^{4-2} + 3^{2-2} = 10 = 10*1$

Hence, x = 1 and z = 10 so, (1, 3, 10) is a solution.

If k > 1, then $3 | 271^{x-1} + 271^{x-2} + \ldots + 271^2 + 271 + 1$

Since all terms are 1(mod 3) so 3|x,

let x = 3m

$(271^m)^3 + (9^k)^3 = z^3$, which is not possible by Lemma 2.2 or Fermat's last theorem represents no solution would be appear for this case.

Case 4.2: Let y = 3k+1 for equation (4), then

$$271^x + 9^{3k+1} = z^3 \quad (4.2.1)$$

here, we consider two cases for k, either it has the value k > 0 or at, k=0

Subcase 4.2 (A): k > 0

$9^{3k+1} \equiv 0 \pmod{81}$

Let x = 2m+1 and z = 2t,

$\because$ x is odd and z is even

Then equation (4.2.1) becomes

$271^{2m+1} + 81 \cdot 9^{3k-1} = 8t^3$

$271^{2m+1} + 271 \cdot 9^{3k-1} - 190 \cdot 9^{3k-1} = 8t^3$

$271(271^{2m} + 9^{3k-1}) = 2(4t^3 + 95 \cdot 9^{3k-1})$

$$((271^{2m} + 9^{3k-1})/2) = ((4t^3 + 95 \cdot 9^{3k-1})/271) = a \quad \text{(say)} \quad (4.2.A.1)$$

from 1 & 3 ratio of equation (4.2.A.1)

$271^{2m} + 9^{3k-1} = 2a$

$$\Rightarrow \quad 9^{3k-1} = 2a - 271^{2m} \quad (4.2.A.2)$$

Also consider from 2 & 3 ratio of equation (4.2.A.1)

$271a = 4t^3 + 95 \cdot 9^{3k-1}$

$= 4t^3 + 190a - 95 \cdot 271^{2m}$ by (4.2.A.2)

$81a = 4t^3 - 95 \cdot 271^{2m}$

$0 \equiv 4t^3 - 14 \cdot 28^{2m} \pmod{81}$

$$\text{or} \quad 0 \equiv 4t^3 - 14 \cdot 55^m \pmod{81} \quad (4.2.A.3)$$

We construct the tables to find the value for different m and t, of $14 \cdot 55^m$ and $4t^3$ modulo 81.

Table -1: To Find all the possible values of $14{\cdot}55^m$ (mod 81)

Find all the possible values of $14{\cdot}55^m$ (mod 81)		
m	55^m (mod 81)	$14{\cdot}55^m$ (mod 81)
1	55	41
2	28	68
3	1	**14**
4	55	41

for m > 3, repeat the values under (mod 81).

Table -2: To Find all the possible values of $4t^3$ (mod 81)

Find all the possible values of $4t^3$ (mod 81)		
t	t^3(mod 81)	$4t^3$(mod 81)
0	0	0
1	1	4
2	8	32
3	27	27
4	64	13
5	44	**14**
6	54	54
7	19	76
8	26	23
9	0	0

for t > 8, repeat the values under (mod 81)

we can compare these values of t and m, according to the congruence (4.2.A.3) then here, z = 2t and $t \equiv 5 (\text{mod } 9)$ because t > 8, repeat the values under (mod 81)

$$z \equiv 1(\text{mod } 9) \text{ and } m \equiv 0(\text{mod } 3)$$

$$\Rightarrow \quad z^3 \equiv 1(\text{mod } 81) \text{ and } x \equiv 1(\text{mod } 3).$$

Let x = 3n+1

Since $z^3 \equiv 1(\text{mod } 81)$ and $9^{3k+1} \equiv 0(\text{mod } 81)$, from equation (4.2.1)

$$\Rightarrow \quad 271^x \equiv 1(\text{mod } 81)$$

$$\Rightarrow \quad 271{\cdot}(271^3)^n \equiv 1 \ (\text{mod } 81)$$

While,

$$271 \equiv 28(\text{mod } 81)$$

which is a contradiction as 1 and 28 are not congruent to each other under modulo 81. So, the equation (4.2.1) has no solution for this case.

Subcase 4.2 (B): k = 0

Let the equation (4.2.1) at, k=0

$$271^x + 9 = z^3 \qquad (4.2.B.1)$$

We consider different modulo for the equation (4.2. B.1), First consider modulo 5, then

$$271^x + 9 \equiv z^3 \ (\text{mod } 5)$$

$$271^x + 9 \equiv 0 \ (\text{mod } 5), \qquad [\because 271 \equiv 1 \ (\text{mod } 5)]$$

$$\Rightarrow \quad z \equiv 0 \ (\text{mod } 5)$$

$$\Rightarrow \quad z^3 \equiv 0 \ (\text{mod } 25)$$

From, (4.2. B.1), consider modulo 25, then

$$\Rightarrow \quad 21^x \equiv 16 \ (\text{mod } 25),$$

$$\because \quad 271 \equiv 21 \ (\text{mod } 25)$$

271^x can be 21, 16, 11, 6, 1 (mod 25), if x takes the value from 1, 2, 3, 4 and 5 resp. so, here we see x > 5, it will repeat the values.

Hence $x \equiv 2 \ (\text{mod } 5)$

Now from equation (4.2. B.1),

$$271^x + 9 \equiv z^3 \ (\text{mod } 31)$$

$$23^x + 9 \equiv z^3 \ (\text{mod } 31)$$

(23^x +9) can be 1, 11, 24, 13, 8, 17, 7, 25, 5,10 (mod 31), if x takes the value from 1 to 10 respectively and more. So, we take x > 10, it will repeat the values.

$$\because \quad x \equiv 2, 7 \ (\text{mod } 10)$$

Hence, $z^3 \equiv 7,11(\text{mod } 31)$. Which contradicts, because z^3 can never takes the value 7 or 11 under (mod 31), so subcase 4.2 (B) unable to show the solution at k=0.

Case 4.3: y = 3k+2, then equation (4) becomes

Let $$271^x + 9^{3k+2} = z^3 \qquad (4.3.1)$$

$$9^{3k+2} \equiv 0(\text{mod } 81)$$

Let x = 2m+1 and z = 2t [∵ x is odd and z is even]

Then equation (4.3.1) becomes

$$271^{2m+1} + 81{\cdot}9^{3k} = z^3$$

$$271(271^{2m} + 9^{3k}) = 2(4t^3 + 95{\cdot}9^{3k})$$

To factor out 542 because 271 is a factor of 542, Since both side of the above equation is divisible by 542, let it be 542a then,

$$271^{2m} + 9^{3k} = 2a$$

$$\Rightarrow \quad 9^{3k} = 2a - 271^{2m}$$

Also, $271\, a = 4t^3 + 95{\cdot}9^{3k}$

$$= 4t^3 + 95(2a - 271^{2m})$$

$= 4t^3 + 190a - 95 \cdot 271^{2m}$

$81a = 4t^3 - 95 \cdot 271^{2m}$

Using subcase 4.2 (A); it has no solution.

Hence the equation $271^x + 9^y = z^3$ has a solution (1, 3, 10).

5. CONCLUSIONS

We have examined the equation $512^x + 1728^y = z^3$ has no solution in Z_+ and also obtained the solution of the non-linear exponential Diophantine equation $271^x + 9^y = z^3$, where (x, y, z) = (1, 3, 10) are non-negative integers.

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