International Research Journal of Engineering and Technology (IRJET) e-ISSN: 2395-0056
Volume: 10 Issue: 04 | Apr 2023 www.irjet.net p-ISSN: 2395-0072

Solution of Ordinary Differential Equation with Initial Condition Using New Elzaki Transform

Sunil Shrivastava[1], Dr. Kalpana Saxena[2]

[1]Assistant professor, Department of Mathematics, Rabindra nath Tagore University, Bhopal(M.P.)
[2]Proffessor, Department of Mathematics, Govt. Motilal Vigyan Mahavidyalya, Bhopal(M.P.)

Abstract- *In this paper the solution of Ordinary Differential Equation with initial condition by using the new Elzaki transform is introduced. The new Elzaki transform is modified version of Laplace and Sumudu transform, and it also expand the nth order derivatives by mathematical induction method. Also in this paper we have explained the properties of Elzaki transform, with inversion form of the transform. With this application we can generate simple formula for solving First order first degree, and Second order first degree ordinary differential equations, with constant coefficients.*

Keywords: *Elzaki transform Ordinary differential equations.*

AMS Subject Classification: 44AXX, 34A30, 34A12.

1. INTRODUCTION

Ordinary differential equations are used in many areas of engineering and basic science like application Beams, Electrical circuits, Dynamics, etc the Laplace transform are some of the well known ordinary differential equations used in these fields. Many type of Ordinary differential equations also can be solved with aid of integral transforms such as Laplace transform, Fourier transform, Sumudu transform[1,5]. In this paper, we have studied to obtain a formula for a special solution of in the most general case nth order Ordinary Differential Equations with constant coefficient. Also found the solution of first order first degree & second order first degree linear differential equation with constant coefficient [3,4].

The Elzaki transform method used in several areas of mathematics is an integral transform. We can solve linear differential equation with use Elzaki transform operator moreover partial differential equations, integral equations & integro differential equations. This method can not be suitable for solution of non-linear differential equations. However non-linear differential equation can be solved by using Elzaki transform aid with differential transform method.

Elzaki transform define for function $f(t)$ of exponential order, function $f(t)$ define with a set A below,

$$A = \left\{ f(t): \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, if\ t \in (-1)^j in\ \times [0,\infty) \right\}$$

Hear constant M must be real finite number and k_1, k_2 may be finite or infinite.

Elzaki transform denoted by operator E,

$$E\{f(t)\} = v\int_0^\infty f(t)e^{-\frac{t}{v}}dt = T(v),\ t \ge 0,\ \ k_1 \le v \le k_2$$

2. ELZAKI TRANSFORM OF SOME FUNCTIONS

$f(t)$	$E\{f(t)\} = T(v)$	$f(t)$	$E\{f(t)\} = T(v)$
1	v^2	$cosat$	$\frac{av^2}{1+a^2v^2}$
t	v^3	$sinhat$	$\frac{av^3}{1-a^2v^2}$
t^n	$n!\,v^{n+2}$	$coshat$	$\frac{av^2}{1-a^2v^2}$
$\frac{t^{a-1}}{\Gamma a}\ a > 0$	v^{a+1}	$e^{at}\ sinbt$	$\frac{bv^3}{(1-av)^2+b^2v^2}$
e^{at}	$\frac{v^2}{1-av}$	$e^{at}\ cosbt$	$\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$
te^{at}	$\frac{v^3}{(1-av)^2}$	$t\ sinat$	$\frac{2av^4}{1+a^2v^2}$

$\frac{t^{n-1}e^{at}}{(n-1)!}$ $n = 1,2,3 \ldots$	$\frac{v^{n+1}}{(1-av)^n}$	$J_0(at)$	$\frac{v^2}{\sqrt{1+a^2v^2}}$
$sinat$	$\frac{av^3}{1+a^2v^2}$	$H(t-a)$	$v^2e^{-\frac{a}{v}}$

Elzaki transform of first order derivatives of $y = f(t)$ *as following:*

$$E\left\{\frac{dy}{dt}\right\} = \frac{1}{v}T(v) - vf(0)$$

Elzaki transform of second order derivatives of $y = f(t)$ *as following:*

$$E\left\{\frac{d^2y}{dt^2}\right\} = \frac{1}{v^2}T(v) - f(0) - v\frac{d}{dt}f(0)$$

Elzaki transform of third order derivatives of $y = f(t)$ *as following:*

$$E\left\{\frac{d^3y}{dt^3}\right\} = \frac{1}{v^3}T(v) - \frac{1}{v}f(0) - \frac{d}{dt}f(0) - v\frac{d^2}{dt^2}f(0)$$

Now, find the expansion of nth order derivative after apply Elzaki transform and by this expansion I have used some Lemma & examples and introduce the new results for first order first degree (FOFD) & second order first degree (SOFD) ordinary differential equations with constant coefficients.

3. EXPANSION OF NTH ORDER DERIVATIVE WITH ELZAKI OPERATOR

Theorem: 3.1

Elzaki transform of nth order derivatives of $f(t)$*, is .*

$$E\left\{\frac{d^nf}{dt^n}\right\} = \frac{1}{v^n}T(v) - \frac{1}{v^{n-2}}f(0) - \frac{1}{v^{n-3}}\frac{d}{dt}f(0) - - - - - - - \frac{d^{n-2}}{dt^{n-2}}f(0) - v\frac{d^{n-1}}{dt^{n-1}}f(0).$$

Proof:

Let $f(t)$ *any function , for the result taking Mathematical induction method,*

If $n = 1$ *then,*

$$E\left\{\frac{df}{dt}\right\} = \frac{1}{v}T(v) - vf(0)$$

Theorem is true for n=1.

If $n = 2$ *then,*

$$E\left\{\frac{d^2f}{dt^2}\right\} = \frac{1}{v^2}T(v) - f(0) - v\frac{d}{dt}f(0)$$

We assume that the theorem is true for n=k,

$$E\left\{\frac{d^kf}{dt^k}\right\} = \frac{1}{v^k}T(v) - \frac{1}{v^{k-2}}f(0) - \frac{1}{v^{k-3}}\frac{d}{dt}f(0) - - - - - - \frac{d^{k-2}}{dt^{k-2}}f(0) - v\frac{d^{k-1}}{dt^{k-1}}f(0).$$

Now we have to show the theorem is true for n=k+1,

$$E\left\{\frac{d^{k+1}f}{dt^{k+1}}\right\} = v\int_0^\infty e^{-t/v}\frac{d^{k+1}f}{dt^{k+1}}dt$$

$$= v\left[\left\{e^{-t/v}\frac{d^kf}{dt^k}\right\}_0^\infty + \int_0^\infty \frac{1}{v}e^{-t/v}\frac{d^kf}{dt^k}dt\right]$$

$$= v\left[\left\{\lim_{t\to\infty} e^{-t/v}\frac{d^kf}{dt^k}\right\} - e^0\frac{d^k}{dt^k}f(0) + \frac{1}{v}\int_0^\infty e^{-t/v}\frac{d^kf}{dt^k}dt\right]$$

$$= \frac{1}{v}E\left\{\frac{d^kf}{dt^k}\right\} - v\frac{d^k}{dt^k}f(0)$$

Now,

$$= \frac{1}{v}\left[\frac{1}{v^k}T(v) - \frac{1}{v^{k-2}}f(0) - \frac{1}{v^{k-3}}\frac{d}{dt}f(0) - - - - - - \frac{d^{k-2}}{dt^{k-2}}f(0) - v\frac{d^{k-1}}{dt^{k-1}}f(0)\right] - v\frac{d^k}{dt^k}f(0)$$

$$\Rightarrow E\left\{\frac{d^{k+1}f}{dt^{k+1}}\right\} = \frac{1}{v^{k+1}}T(v) - \frac{1}{v^{k-1}}f(0) - \frac{1}{v^{k-2}}\frac{d}{dt}f(0) - - \frac{1}{v}\frac{d^{k-2}}{dt^{k-2}}f(0) - \frac{d^{k-1}}{dt^{k-1}}f(0) - v\frac{d^k}{dt^k}f(0)$$

Required result.

Lemma. 3.2

Elzaki solution of First Order First Degree (FOFD) Linear Differential Equations.

$$A\frac{dy}{dt} + By = f(t), \quad initial\ condition, \quad y(0) = p$$

A, B are the constant.

Proof:

Given First Order First Degree (FOFD) Linear Differential Equations,

$$A\frac{dy}{dt} + By = f(t), \quad y(0) = p$$

Taking Elzaki transform both side,

$$AE\left\{\frac{dy}{dt}\right\} + BE\{y\} = E\{f(t)\}$$

By theorem 1,

$$\Rightarrow A\left\{\frac{1}{v}T(v) - vf(0)\right\} + BE(y) = E\{f(t)\}$$

$$Since\ E(y) = T(v)$$

$$\Rightarrow \frac{A}{v}E(y) - Avp + BE(y) = E\{f(t)\}$$

$$\Rightarrow \left\{\frac{A}{v} + B\right\}E(y) = Avp + E\{f(t)\}$$

$$\Rightarrow E(y) = \left[\frac{v\{Avp + Ef(t)\}}{A + Bv}\right]$$

$$y = E^{-1}\left[\frac{v\{Avp + Ef(t)\}}{A + Bv}\right]$$

It is the solution of FOFD linear differential equation.

Lemma. 3.3

Elzaki solution of Second Order First Degree (SOFD) Linear Differential Equations.

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = f(t)\ \ initial\ conditions\ y(0) = p, y'(0) = q$$

$$A, B\ \&\ C\ are\ the\ constant.$$

Proof:

Given Second Order First Degree (SOFD) Linear Differential Equations.

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = f(t)\ \ initial\ conditions\ y(0) = p, y'(0) = q$$

Taking Elzaki transform,

$$AE\left\{\frac{d^2y}{dt^2}\right\} + BE\left\{\frac{dy}{dt}\right\} + CE\{y\} = E\{f(t)\}$$

By theorem 1,

$$\Rightarrow A\left\{\frac{1}{v^2}T(v) - f(0) - v\frac{d}{dt}f(0)\right\} + B\left\{\frac{1}{v}T(v) - vf(0)\right\} + CE(y) = E\{f(t)\}$$

$$\Rightarrow \frac{A}{v^2}E(y) - Ap - Avq + \frac{B}{v}E(y) - Bvp + CE(y) = E\{f(t)\}$$

$$\Rightarrow \left\{\frac{A}{v^2} + \frac{B}{v} + C\right\}E(y) = Ap + Avq + Bvp + E\{f(t)\}$$

$$E(y) = \left[\frac{v^2\{Ap + Avq + Bvp + Ef(t)\}}{A + Bv + Cv^2}\right]$$

$$y = E^{-1}\left[\frac{v^2\{Ap + Avq + Bvp + Ef(t)\}}{A + Bv + Cv^2}\right]$$

It is the solution of SOFD linear differential equation.

Example 3.2.1

$$First\ order\ first\ degree\ differential\ equation\ \frac{dy}{dt} + y = 0, y(0) = 1.$$

By the Lemma 3.2,

$$y = E^{-1}\left[\frac{v\{Avp + Ef(t)\}}{A + Bv}\right]$$

$$Hear, A = 1, B = 1, p = 1\ \&\ f(t) = 0$$

$$y = E^{-1}\left[\frac{v\{v + 0\}}{1 + v}\right]$$

$$y = E^{-1}\left[\frac{v^2}{1 + v}\right]$$

$y(t) = e^{-t}$ Required result.

Example 3.2.2

$$First\ order\ first\ degree\ differential\ equation\ \frac{dy}{dt} - 4y = 1, y(0) = 1$$

By the Lemma 3.2,

$$y = E^{-1}\left[\frac{v\{Avp + Ef(t)\}}{A + Bv}\right]$$

$$Hear, A = 1, B = -4, p = 1\ \&\ f(t) = 1$$

$$y = E^{-1}\left[\frac{v\{v + E(1)\}}{1 - 4v}\right]$$

$$y = E^{-1}\left[\frac{v\{v + v^2\}}{1 - 4v}\right]$$

$$y = E^{-1}\left[\frac{v^2}{1 - 4v}\right] + E^{-1}\left[\frac{v^3}{1 - 4v}\right]$$

After inversion,

$$y = \frac{5}{4}e^{4t} - \frac{1}{4}$$

Required result.

Example 3.3.1

$$Second\ order\ first\ degree\ differential\ equation\ \frac{d^2y}{dt^2}$$

$$-3\frac{dy}{dt}+2y=0, y(0)=1, y'(0)=4$$

By the Lemma 3.3,

$$y=E^{-1}\left[\frac{v^2\{Ap+Avq+Bvp+Ef(t)\}}{A+Bv+Cv^2}\right]$$

$$Hear, A=1, B=-3, C=2, p=1, q=4\ \&\ f(t)=0$$

$$y=E^{-1}\left[\frac{v^2\{1+4v-3v+0\}}{2v^2-3v+1}\right]$$

$$y=E^{-1}\left[\frac{v^2(1+v)}{(2v-1)(v-1)}\right]$$

$$y=3E^{-1}\left\{\frac{v^2}{1-2v}\right\}-2E^{-1}\left\{\frac{v^2}{1-v}\right\}$$

After inversion,

$$y=3e^{2t}-2e^{t}$$

Required result.

Example 3.3.2

$$Second\ order\ first\ degree\ differential\ equation\ \frac{d^2y}{dt^2}$$

$$+4\frac{dy}{dt}+3y=e^t, y(0)=0, y'(0)=2$$

By the Lemma 3.3,

$$y=E^{-1}\left[\frac{v^2\{Ap+Avq+Bvp+Ef(t)\}}{A+Bv+Cv^2}\right]$$

$$Hear, A=1, B=4, C=3, p=0, q=2\ \&\ f(t)=e^t$$

$$y=E^{-1}\left[\frac{v^2\{0+2v+0+E(e^t)\}}{1+4v+3v^2}\right]$$

$$y=E^{-1}\left[\frac{v^2\left\{2v+\frac{v^2}{1-v}\right\}}{1+4v+3v^2}\right]$$

$$y=E^{-1}\left[\frac{v^2(2v-v^2)}{(1-v)(1+3v)(1+v)}\right]$$

$$y=\frac{1}{8}E^{-1}\left\{\frac{v^2}{1-v}\right\}+\frac{3}{4}E^{-1}\left\{\frac{v^2}{1+v}\right\}-\frac{7}{8}E^{-1}\left\{\frac{v^2}{1+3v}\right\}$$

After inversion,

$$y=\frac{1}{8}e^{t}+\frac{3}{4}e^{-t}-\frac{7}{8}e^{-3t}$$

Required Result.

4. CONCLUSION

In this paper we have applied Elzaki transform with the new formula on ordinary differential equation of first order first degree and second order first degree with the result. This formula is also applicable for nth order, it is very useful and effective method. This process is also very useful for other type of differential equations, applied mathematics, engineering and science.

5. REFERENCES

[1] Elzaki, T.M., and Ezaki, S.M., On the connections between Laplace and ELzaki transforms, Advance in Theoretical and Applied Mathematics, 6(1), pp.1-11 (2011)

[2] Elzaki, T.M., The new integral transform Elzaki transform, Global Journal of Pure and Applied Mathematics 2011, 7(1), pp. 57-64 (2011)

[3] Elzaki, T.M., On the new integral transform Elzali transform fundamental properties investigation and applications, Global Journal of Mathematical Science: Theory and Practical 2012, 4(1), pp. 1-13 (2012)

[4] Elzaki, T.M., and Ezaki, S.M , On the Elzaki transform and higher order ordinary differential equations, Advance in Theoretical and Applied Mathematics, 6(1), pp. 107-113 (2011)

[5] Elzaki, T.M., Ezaki, S.M and Eman, M.A.H., Elzaki and Sumudu Transform for solving some differential equations, Global Journal of Pure and Applied Mathematics, 8(2), pp. 167-173 (2012)

[6] Elzaki, T. M., Solution of non-linear differential equations using mixture of Elzaki transform and differential transform method, International Mathematical Forum, Volume , 7(13), pp. 631-638 (2012)

[7] Elzaki, T.M., and Ezaki, S.M , On The connections between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics , 6(1), pp 1-10 (2012) .

[8] Elzaki, T.M. and Kim, H. , The solution of radial diffusivity and shock wave equations by Elzaki variational iteration method, International Journal of Mathematical Analysis , 9(21), pp. 1065–1071 (2015).

[[9] Kim, H., The intrinsic structure and properties of Laplace-typed integral transforms, Hindawi

Mathematical Problems in Engineering , Article ID 1762729, pp. 8 (2017).

[[10] Hossain, Md. B. and Dutta, M., Solution of linear partial differential equations with mixed partial derivatives by Elzaki substitution method, American Journal of Computational and Applied Mathematics 2018, 8(3), pp. 59-64 (2018).

[[11] Duzi, M. and Elzaki, T.M., Solution of constant coefficients partial derivative equations with Elzaki transform method, TWMS J. App. Eng. Math, 9(3) , pp. 563-570 (2019).

[[12] Sharjeel, S. and Barakzai, M.A.K., Some new applications of Elzaki transform for solution of Volterra type integral equations, Journal of Applied Mathematics and Physics 2019, 7, pp. 1877-1892 (2019).