Cauchy's Inequality based study of the Differential Equations and the Simple Transformation Techniques for Algebra

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Abstract – *This paper instigates the multivariate generalization of the widely used Cauchy inequality $1 + x \leq e^x$ where x can be any non-negative real number. The results of this study can be the solution for the Cauchy's problem for particular Ordinary Differential Equation (ODE). This is also related to study of the complete monotone function and the divided differences theory. The proof is based on the empty product convention notion and the Beppo Levi theorem of Monotone convergence. This study is also extended to multivariate generalization of the simultaneous inequalities.*

Key Words: ODEs, Inequalities, Population Dynamics, Simultaneous Inequalities, Divided differences.

1.INTRODUCTION

The elemental inequality that is being used for various applications in Plant Biology, Olympiad Inequality Problems , Image Processing, Signal processing and various computer application is given in (1).

$$1 + x \leq e^x \qquad (1)$$

Where x can be any non-negative real number, generally x is considered as smaller values in real time applications. However, multivariate generalization of (1) is not established. This is the main focus of this paper is to prove (2) specifically.

$$\prod_{i=1}^{n}(1+x_k)^{a_k} \leq e^{\frac{1}{m}\prod_{i=1}^{n}x_k} \qquad (2)$$

Where $x_1, x_2, \ldots, x_n$ are the pairwise non-negative distinct real numbers. Here

$$a_k := \frac{\prod_{l=1\, l\neq k}^{n} x_l}{\prod_{l=1\, l\neq k}^{n}(x_l - x_k)}$$

The empty product convention is made, so that (2) can be changed as (1) when the n value is 1. In this paper, we also show that the inequality in (2) is only correct when the value of x_k is 0. (1), (2) are extended into whole Euclidean space when n value is 2, and (2) is indeterminate for in the range of (2,-2), (0,-1) and (1,-1/2) or can also be considered as (-1/4, -1/2).

From the analogue of specific ODE Cauchy problem, the generalized inequality form (2) is defined. This ODE Cauchy problems are widely used in the plant biology for chromosome analysis, population dynamics, to predict the virus mutations and many other problems. A direct way of solution is required to handle these real-life inequality-based problems. However, this cannot be analysed using the elementary methods since the problems are complex in nature. The monotonic study of functions and mean value theorem of divided difference of functions are essential for complex problems. In next section of this paper ODE approach of solving i.e., the study of Cauchy problem is shown the with the aforementioned solutions. The straightforward approach of preliminary analysis of the given inequality is presented in the section 3 of the paper. The conclusion along with future scope of the work is generalized in the section 4.

2. ODE Approach of solving

2.1 Primary Analysis

In this paper, autonomous ODE Cauchy problem is considered as given in (3).

$$\begin{cases} \frac{dy}{dk}(k) = f\left(y(k) := (-1)^{m+1} y(k) \prod_{i=1}^{m}\left(1 - \frac{1}{t_i} y(k)\right)\right. \\ y(0) = y_0 \end{cases} \qquad (3)$$

Where $t_1, t_2, \ldots, t_n$ are the distinct positive real numbers (pairwise) in the increasing order. The convential representations of (3) when n is 1 and n is 2 are the standard logistic model and the standard logistic model with the Alle effect. The eccentric maximal smooth solution can be easily deduced by the concerning classic theory of the Cauchy problems of ODEs from the problems data as (4) and the phase line is depicted in Fig.- 1.

$$y: (-\varepsilon_a, \varepsilon_b) \to \mathbb{R} \quad \text{where } \varepsilon_{a,b} \in (0, \infty] \qquad (4)$$

The y in the Fig 1 is,

$$\begin{cases} global\ negative, & when\ y_0 < 0\ , n\ \equiv 1\ mod\ 2 \\ global\ positive, & when\ y_0 < 0\ , n\ \equiv 1\ mod\ 2 \\ global\ positive, & when\ y_0 > t_n \\ global, when\ y_0 \in [0, t_n] & \end{cases}$$

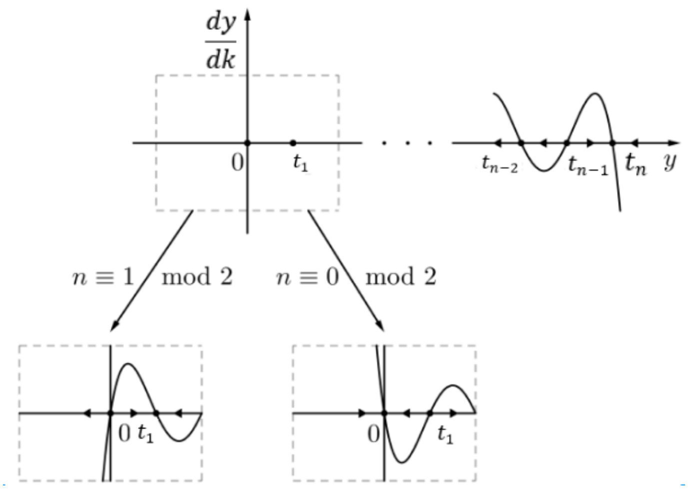

Fig-1. The phase line graph of (3)

For the uniqueness of the time invariant sets are utilized,

$(-\infty, 0)$, null, $(0, t_1), t_1, (t_1, t_2), \ldots\ldots, (t_{n-1}, t_n), t_n$.

The behavior of y at various n values is as follows

1. $y_0 < 0\ and\ \begin{cases} k \to \varepsilon_b, when\ n\ \equiv 1\ mod\ 2 \\ k \to \varepsilon_a, when\ n\ \equiv 0\ mod\ 2 \end{cases}$
2. $y_0 > t_n\ and\ k \to \varepsilon_a$

to conclude the qualitative analysis.

Now, there exists two cases if $y_0 < 0$

Case (1):

a. If the value of n is $n\ \equiv 1\ mod\ 2$, then y(k) <0 and f(y(k)) <0 when t $\in (\varepsilon_a, \varepsilon_b)$, y is decreasing monotonically then the k is

$$k = \int_0^k dx = \int_{y(k)}^{y_0} \left|\frac{1}{f(x)}\right| = \int_{y(k)}^{y_0} -\frac{1}{f(x)}\, dx$$

$$= \int_{y(k)}^{y_0} -\frac{1}{(-1)^{n+1}\prod_{i=1}^{n}(1-\frac{1}{t})}\, dx$$

$$= \int_{y(k)}^{y_0} \frac{\prod_{i=1}^{n} t_i}{-x\prod_{i=1}^{n}(t_i - x)}\, dx\ , where\ \mathrm{k} \in (0, \varepsilon_b)$$

Hence,

$$\varepsilon_b = \int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{x\prod_{i=1}^{n}(t_i - x)}\, dx$$

$$= \int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{\prod_{i=1}^{n}(\ 1 - \frac{t_i}{x}\ )}\, dx \qquad (5)$$

Here y is the maximal and the inequalities are as follows

$$\frac{\prod_{i=1}^{n} t_i}{(-x)^{n+1}} > 0 \text{ and } 0 < \frac{1}{\prod_{i=1}^{n}(\ 1 - \frac{t_i}{x}\ )} < 1 \text{ where } x < y_0 \qquad (6)$$

The bound is achieved by

$$\varepsilon_b < \int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{(-x)^{n+1}}\, dx = \frac{\prod_{i=1}^{n} t_i}{n\ (-y_0)} < \infty \qquad (7)$$

The value of y increases significantly and becomes the problem of exploding gradient arises.

b. If the value of n is $n\ \equiv 0\ mod\ 2$, then y(k) <0 and f(y(k)) > 0 when t $\in (-\varepsilon_a, \varepsilon_b)$, y is increasing monotonically then the k is

$$-k = \int_t^0 dx = \int_{y(k)}^{y_0} \left|\frac{1}{f(x)}\right| = \int_{y(k)}^{y_0} \frac{1}{f(x)}\, dx$$

$$= \int_{y(k)}^{y_0} -\frac{1}{(-1)^{n+1}\prod_{i=1}^{n}(1 - \frac{1}{t}x)}\, dx$$

$$= \int_{y(k)}^{y_0} \frac{\prod_{i=1}^{n} t_i}{-x\prod_{i=1}^{n}(t_i - x)}\, dx\ , where\ \mathrm{k} \in (\varepsilon_a, 0)$$

Hence,

$$\varepsilon_a = \int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{x\prod_{i=1}^{n}(t_i - x)}\, dx$$

$$= \int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{(-x)^{n+1}\prod_{i=1}^{n}(\ 1 - \frac{t_i}{x}\ )}\, dx \qquad (8)$$

Here y is the maximal and the inequalities are as follows

$$\varepsilon_a < \int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{(-x)^{n+1}}\, dx = \frac{\prod_{i=1}^{n} t_i}{n\ (-y_0)^n} < \infty \qquad (9)$$

The value of y increases significantly and becomes the problem of exploding gradient arises.

Case (2):

If the value of $y_0 > t_n$, then y(k) > t_n ,giving f(y(t)) < 0 where t $\in (\varepsilon_a, \varepsilon_b)$. And the value of y is decreasing monotonically.

$$-t = \int_t^0 dx = \int_{y_0}^{y(k)} \left|\frac{1}{f(x)}\right| dx = \int_{y_0}^{y(k)} -\frac{1}{f(x)} dx$$

$$= \int_{y_0}^{y(k)} -\frac{1}{(-1)^{n+1}\prod_{i=1}^{n}(1-\frac{1}{t})} dx \text{ where k} \in (\varepsilon_a, 0)$$

Hence,

$$\varepsilon_a = \int_{y_0}^{\infty} -\frac{\prod_{i=1}^{n} t_i}{x\prod_{i=1}^{n}(x-t_i)} dx$$

$$= \int_{y_0}^{\infty} -\frac{\prod_{i=1}^{n} t_i}{x^{n+1}\prod_{i=1}^{n}(1-\frac{t_i}{x})} dx$$

Here y is the maximal and the inequalities are as follows

$\frac{\prod_{i=1}^{n} t_i}{(x)^{n+1}} > 0$ and $0 < \frac{1}{\prod_{i=1}^{n}(1-\frac{t_i}{x})} < \frac{1}{\prod_{i=1}^{n}(1-\frac{k_i}{y_0})}$ where $x > y_0$

The bound is achieved by

$$\varepsilon_a < \int_{y_0}^{\infty} \frac{\prod_{i=1}^{n} t_i}{(x)^{n+1}\prod_{i=1}^{n}(1-\frac{t_i}{y_0})} dx = \frac{\prod_{i=1}^{n} t_i}{n\,(y_0)^n \prod_{i=1}^{n}(1-\frac{t_i}{y_0})} < \infty \quad (10)$$

The value of y increases significantly and becomes the problem of exploding gradient arises.

2.2 Final result derivation

From the equations (5), (7), (9), (10) , the further calculations is computed to get the blow up time exactly of extreme solution where $y_0 < 0$ and $y_0 > t_n$.

The aforementioned relations of the functions inside the are analyzed further:

1. Decomposition of the partial fractions is adopted

$$\frac{\prod_{i=1}^{n} t_i}{-x\prod_{i=1}^{n}(t_i-x)} = \frac{L}{-x} + \sum_{i=1}^{n} \frac{L_i}{t_i-x} \text{ where } x < y_0$$

The coefficients $L_1, L_2, L_3, \ldots\ldots, L_n$ are calculated as:

$$\prod_{i=1}^{n} t_i = L_i \prod_{i=1}^{n}(t_i-x) - x\sum_{i=1}^{n} L_i \prod_{j=1\, j\neq i}^{n}(t_i-x) \quad x \in \mathbb{R} \quad (11)$$

When x=0 we get L=1 in (11). The coefficient of x^n in the RHS of (11) is eliminated to get:

$$\prod_{i=1}^{n} t_i = -t_{i0}\sum_{i=1}^{n} L_i \prod_{j=1\, j\neq i0}^{n}(t_i - t_{i0})$$

$$= -t_{i0}L_{i0} \prod_{j=1\, j\neq i0}^{n}(t_i - t_{i0})$$

$$\Rightarrow L_i = \frac{\prod_{j=1\, j\neq i0}^{n} t_j}{\prod_{j=1\, j\neq i0}^{n}(t_j - t_i)}$$

2. The analogous matter can be modified as:

$$\frac{\prod_{j=1\, j\neq i0}^{n} t_i}{x\prod_{j=1\, j\neq i0}^{n}(x-t_i)} = \frac{M}{x} + \sum_{i=1}^{n} \frac{M_i}{x-t_i}\ ;\ x > y_0 > t_n$$

Its consequent equality is defined as:

$$M = (-1)^n, M_i = \frac{\prod_{j=1\, j\neq i0}^{n} t_i}{-x\prod_{j=1\, j\neq i0}^{n}(t_i-t_j)}$$

$$= (-1)^n L_i \text{ and } \sum_{i=1}^{n} M_i = (-1)^n \sum_{i=1}^{n} L_i = (-1)^{n+1}$$

The values of $\varepsilon_a, \varepsilon_b$ respectively:

1. If $y_0 < 0$, then

$$= \int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{-x\prod_{i=1}^{n}(t_i-x)} dx$$

$$= \int_{-y_0}^{\infty} \frac{1}{x} + \sum_{i=1}^{n} \frac{L_i}{t_i-x} dx$$

$$= \lim_{n\to\infty} \log(x\prod_{i=1}^{n}(t_i+x)^{L_i}) - \log(-y_0\prod_{i=1}^{n}(t_i-y_0)^{L_i})$$

Along with that,

$$\log(x\prod_{i=1}^{n}(t_i+x)^{L_i}) = \log(\prod_{i=1}^{n}(\frac{t_i}{x}+1)^{L_i})$$

Hence, the conclusion is

$$\int_{-\infty}^{y_0} -\frac{\prod_{i=1}^{n} t_i}{x\prod_{i=1}^{n}(t_i-x)} dx = \log(\prod_{i=1}^{n}(\frac{t_i}{-y_0}+1)^{-L_i}) \quad (12)$$

Therefore

$$\log(\prod_{i=1}^{n}(\frac{t_i}{-y_0}+1)^{-L_i}) = \begin{cases}\varepsilon_a & n \equiv 1 \bmod 2\\ \varepsilon_b & n \equiv 0 \bmod 2\end{cases}$$

2. If $y_0 > t_n$, then

$$\log(\prod_{i=1}^{n}(\frac{t_i}{-y_0}+1)^{-L_i}) < \frac{\prod_{i=1}^{n} t_i}{n{y_0}^n\prod_{i=1}^{n}(t_i-x)}$$

Here the value of $x_i := \frac{\frac{t_i}{y_0}}{1-\frac{t_i}{y_0}}$

3. When the x_i value is positive, desired result is achieved.

The equality in (2) only holds when the x values is zero.

3. Direct Approach

Firstly, the definition of the monotone functions is given

3.1 Definition: The function f in the range of (0,∞) is monotonic if and only if the following the condition is satisfied

$$(-1)^y f^y(m) \geq 0 \; where \; (y, m) \in (0, \infty)$$

From the (2), we adopt the classic Beppo Levi theorem and dominated convergence is noted from the Lebesgue theorem, deriving

$$(-1)^y \lim_{y \to 0^+} f^y(m) = y! \int_0^1 k_n dk = \frac{y!}{y+1}$$

The generalization of the result to the higher derivatives is the divided differences of the mean value theorem,

Hypothesis 1: If there are pairwise distinct real number a_1, a_2……., a_n, following the condition

$$k := min\{y\}_j \; and \; K := max\, \{y\}_j \; where \; j \in \mathbb{N}$$

and

$f \in C(k, K) \cap C^{k-1}(k, K)$, then $a_0 \in (k, K)$

$$[a_1, a_2……., a_n, f] := \sum_{i=1\,;\, i \neq j}^{n} \frac{f(a_i)}{a_i - a_j} = \frac{f^{n-1} a_0}{(n-1)!}.$$

3.1 Proof of the result

Assuming that a_1, a_2……., a_n are pairwise distinct positive real numbers and by deducing,

$$\prod_{i=1}^{n} (1 + a_i)^{k_i} < e^{\frac{1}{k}} n \prod_{i=1}^{n} a_i$$

Similarly,

$$\sum_{i=1}^{n} \frac{f(a_i)}{\prod_{j=1\; j \neq i}^{n} (a_i - a_j)} < \frac{1}{n}$$

This can be further analyzed as

$$\sum_{i=1}^{n} \frac{f(a_i)}{\prod_{j=1\; j \neq i}^{n} (a_i - a_j)} = (-1)^{n+1} [a1, a2 \dots \dots, an, , f]$$

Here, the f is strictly increasing monotone then the function $(-1)^{n+1} f^{n-1}$, implies that

$$\frac{(-1)^{n+1} f^{n-1}(a_0)}{(n-1)!} = \frac{1}{n}$$

3.2 Allowed Repetitions

The numbers a_1, a_2……., a_n are positive and distinct in this section of the paper the probable repetitions are shown. For the natural numbers N_1, N_2……., N_n this approach is deduced as shown in (13).

$$\prod_{i=1}^{n} \prod_{j=1}^{N} (1 + \rho a_i) \leq e^{\frac{1}{m} \prod_{i=1}^{n} (1+a_i)^{k_i}} \qquad (13)$$

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